6.1 Absolute Extrema

1. As shown on the graph, the absolute maximum occurs at $x_3$; there is no absolute minimum. (There is no functional value that is less than all others.)

2. As shown on the graph, the absolute minimum occurs at $x_1$; there is no absolute maximum. (There is no functional value that is greater than all others.)

3. As shown on the graph, there are no absolute extrema.

4. As shown on the graph, there are no absolute extrema.

5. As shown on the graph, the absolute minimum occurs at $x_1$; there is no absolute maximum.

6. As shown on the graph, the absolute maximum occurs at $x_1$; there is no absolute minimum.

7. As shown on the graph, the absolute maximum occurs at $x_1$; the absolute minimum occurs at $x_2$.

8. As shown on the graph, the absolute maximum occurs at $x_2$; the absolute minimum occurs at $x_1$.

10. $f(x) = x^3 - 3x^2 - 24x + 5; [-3, 6]$

Find critical numbers:

$$f'(x) = 3x^2 - 6x - 24 = 0$$
$$3(x^2 - 2x - 8) = 0$$
$$3(x + 2)(x - 4) = 0$$
$$x = -2 \text{ or } x = 4$$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>23</td>
</tr>
<tr>
<td>-2</td>
<td>33</td>
</tr>
<tr>
<td>4</td>
<td>-75</td>
</tr>
<tr>
<td>6</td>
<td>-31</td>
</tr>
</tbody>
</table>

11. $f(x) = x^3 - 6x^2 + 9x - 8; [0, 5]$

Find critical numbers:

$$f'(x) = 3x^2 - 12x + 9 = 0$$
$$x^2 - 4x + 3 = 0$$
$$(x - 3)(x - 1) = 0$$
$$x = 1 \text{ or } x = 3$$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-8</td>
</tr>
<tr>
<td>1</td>
<td>-4</td>
</tr>
<tr>
<td>3</td>
<td>-8</td>
</tr>
<tr>
<td>5</td>
<td>12</td>
</tr>
</tbody>
</table>

12. $f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 6x + 3; [-4, 4]$

Find critical numbers:

$$f'(x) = x^2 - x - 6 = 0$$
$$(x + 2)(x - 3) = 0$$
$$x = -2 \text{ or } x = 3$$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-4</td>
<td>$-\frac{7}{3}$ $\approx -2.3$</td>
</tr>
<tr>
<td>-2</td>
<td>$\frac{31}{3}$ $\approx 10.3$</td>
</tr>
<tr>
<td>3</td>
<td>$-\frac{21}{2}$ $\approx -10.5$</td>
</tr>
<tr>
<td>4</td>
<td>$-\frac{23}{3}$ $\approx -7.7$</td>
</tr>
</tbody>
</table>

13. $f(x) = \frac{1}{3}x^3 + \frac{3}{2}x^2 - 4x + 1; [-5, 2]$

Find critical numbers:

$$f'(x) = x^2 + 3x - 4 = 0$$
$$(x + 4)(x - 1) = 0$$
$$x = -4 \text{ or } x = 1$$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-4</td>
<td>$\frac{59}{3}$ $\approx 19.67$</td>
</tr>
<tr>
<td>1</td>
<td>$-\frac{7}{6}$ $\approx -1.17$</td>
</tr>
<tr>
<td>-5</td>
<td>$\frac{101}{6}$ $\approx 16.83$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{5}{3}$ $\approx 1.67$</td>
</tr>
</tbody>
</table>
14. $f(x) = x^4 - 32x^2 - 7; [-5, 6]$
   $f'(x) = 4x^3 - 64x = 0$
   $4x(x^2 - 16) = 0$
   $4x(x - 4)(x + 4) = 0$
   $x = 0$ or $x = 4$ or $x = -4$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-5</td>
<td>-182</td>
</tr>
<tr>
<td>-4</td>
<td>-263</td>
</tr>
<tr>
<td>0</td>
<td>-7</td>
</tr>
<tr>
<td>4</td>
<td>-263</td>
</tr>
<tr>
<td>6</td>
<td>137</td>
</tr>
</tbody>
</table>

15. $f(x) = x^4 - 18x^2 + 1; [-4, 4]$
   $f'(x) = 4x^3 - 36x = 0$
   $4x(x^2 - 9) = 0$
   $4x(x + 3)(x - 3) = 0$
   $x = 0$ or $x = -3$ or $x = 3$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-4</td>
<td>-31</td>
</tr>
<tr>
<td>-3</td>
<td>-80</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>-80</td>
</tr>
<tr>
<td>4</td>
<td>-31</td>
</tr>
</tbody>
</table>

16. $f(x) = \frac{8 + x}{8 - x}; [4, 6]$
   $f'(x) = \frac{(8 - x)(1) - (8 + x)(-1)}{(8 - x)^2}$
   $= \frac{16}{(8 - x)^2}$

   $f'(x)$ is never zero. Although $f'(x)$ fails to exist if $x = 8$, 8 is not in the given interval.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>6</td>
<td>7</td>
</tr>
</tbody>
</table>

17. $f(x) = \frac{1 - x}{3 + x}; [0, 3]$
   $f'(x) = \frac{-4}{(3 + x)^2}$

   No critical numbers

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>\frac{1}{3}</td>
</tr>
<tr>
<td>3</td>
<td>\frac{-1}{3}</td>
</tr>
</tbody>
</table>

18. $f(x) = \frac{x}{x^2 + 2}; [0, 4]$
   $f'(x) = \frac{(x^2 + 2)1 - x(2x)}{(x^2 + 2)^2}$
   $= -\frac{x^2 + 2}{(x^2 + 2)^2} = 0$
   $-\frac{x^2 + 2}{(x^2 + 2)^2} = 0$
   $x^2 = 2$

   $x = \sqrt{2}$ or $x = -\sqrt{2}$, but $-\sqrt{2}$ is not in $[0, 4]$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\sqrt{2}$</td>
<td>$\frac{\sqrt{2}}{4}$</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{2}{9}$</td>
</tr>
</tbody>
</table>

19. $f(x) = \frac{x - 1}{x^2 + 1}; [1, 5]$
   $f'(x) = \frac{-x^2 + 2x + 1}{(x^2 + 1)^2}$
   $f'(x) = 0$ when
   $-x^2 + 2x + 1 = 0$
   $x = 1 \pm \sqrt{2}$,

   but $1 - \sqrt{2}$ is not in $[1, 5]$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{2}{13}$</td>
</tr>
<tr>
<td>$1 + \sqrt{2}$</td>
<td>$\frac{\sqrt{2} - 1}{2}$</td>
</tr>
</tbody>
</table>

20. $f(x) = (x^2 - 16)^{2/3}; [-5, 8]$
   $f'(x) = \frac{2}{3}(x^2 - 16)^{-1/3}(2x) = \frac{4x}{3(x^2 - 16)^{1/3}}$
   $f'(x) = 0$ when $4x = 0$
   $x = 0$

   $f'(x)$ is undefined at $x = -4$ and $x = 4$, but $f(x)$ is defined there, so $-4$ and 4 are also critical numbers.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-5</td>
<td>$9^{2/3} \approx 4.327$</td>
</tr>
<tr>
<td>-4</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>$(-16)^{2/3} \approx 6.350$</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>$48^{2/3} \approx 13.21$</td>
</tr>
</tbody>
</table>
21. \( f(x) = (x^2 - 4)^{1/3}; [-2, 3] \)

\[
\begin{align*}
f'(x) &= \frac{1}{3} (x^2 - 4)^{-2/3} (2x) \\
&= \frac{2x}{3(x^2 - 4)^{2/3}} \\
f'(x) &= 0 \text{ when } 2x = 0 \text{ or } x = 0
\end{align*}
\]

\( f'(x) \) is undefined at \( x = -2 \) and \( x = 2 \), but \( f(x) \) is defined there, so \(-2 \) and \( 2 \) are also critical numbers.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>((-4)^{1/3} \approx -1.587) Absolute minimum</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>(5^{1/3} \approx 1.710) Absolute maximum</td>
</tr>
</tbody>
</table>

22. \( f(x) = x + 3x^{2/3}; [-10, 1] \)

\[
\begin{align*}
f'(x) &= 1 + 2x^{-1/3} \\
&= 1 + \frac{2}{\sqrt[3]{x}} \\
&= \sqrt[3]{x} + 2 \\
f'(x) &= 0 \text{ when } \sqrt[3]{x} + 2 = 0 \\
&\sqrt[3]{x} = -2 \\
x &= -8
\end{align*}
\]

\( f'(x) \) is undefined at \( x = 0 \), but \( f(x) \) is defined at \( x = 0 \), so \( 0 \) is also a critical number.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-10</td>
<td>3.925</td>
</tr>
<tr>
<td>-8</td>
<td>4 Absolute maximum</td>
</tr>
<tr>
<td>0</td>
<td>0 Absolute minimum</td>
</tr>
<tr>
<td>1</td>
<td>4 Absolute maximum</td>
</tr>
</tbody>
</table>

23. \( f(x) = 5x^{2/3} + 2x^{5/3}; [-2, 1] \)

\[
\begin{align*}
f'(x) &= \frac{10}{3} x^{-1/3} + \frac{10}{3} x^{2/3} \\
&= \frac{10}{3} x^{1/3} + \frac{10x^{2/3}}{3} \\
&= \frac{10x + 10}{3} x^{1/3} \\
&= \frac{10(x + 1)}{3\sqrt[3]{x}} \\
f'(x) &= 0 \text{ when } 10(x + 1) = 0 \\
x + 1 &= 0 \\
x &= -1
\end{align*}
\]

24. \( f(x) = \frac{\ln x}{x^2}; [1, 4] \)

\[
\begin{align*}
f'(x) &= \frac{x^2 \left( \frac{1}{x} - \ln x \cdot 2x \right)}{x^4} \\
&= \frac{x - 2x\ln x}{x^4} \\
&\frac{x(1 - 2\ln x)}{x^4} \\
&= \frac{1 - 2\ln x}{x^3}
\end{align*}
\]

\( f'(x) = 0 \text{ when } 1 - 2\ln x = 0 \)

\( -2\ln x = -1 \)

\( \ln x = \frac{1}{2} \)

\( x = e^{1/2} \)

Although \( f'(x) \) fails to exist at \( x = 0 \), \( 0 \) is not in the specified domain for \( f(x) \), so \( 0 \) is not a critical number.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0 Absolute minimum</td>
</tr>
<tr>
<td>( e^{1/2} )</td>
<td>0.1839 Absolute maximum</td>
</tr>
<tr>
<td>4</td>
<td>0.0866</td>
</tr>
</tbody>
</table>

25. \( f(x) = x^2 - 8\ln x; [1, 4] \)

\[
\begin{align*}
f'(x) &= 2x - \frac{8}{x} \\
f'(x) &= 0 \text{ when } 2x - \frac{8}{x} = 0 \\
&2x = \frac{8}{x} \\
&2x^2 = 8 \\
x^2 = 4 \\
x = \pm 2 \\
& x = -2 \text{ or } x = 2
\end{align*}
\]

but \( x = -2 \) is not in the given interval.
Although $f'(x)$ fails to exist at $x = 0$, 0 is not in the specified domain for $f(x)$, so 0 is not a critical number.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.00</td>
</tr>
<tr>
<td>2</td>
<td>1.545 Absolute minimum</td>
</tr>
<tr>
<td>4</td>
<td>4.910 Absolute maximum</td>
</tr>
</tbody>
</table>

26. $f(x) = x^2e^{-0.5x}$; [2, 5]

$$f'(x) = x^2 \left(-\frac{1}{2}e^{-0.5x}\right) + 2xe^{-0.5x} = e^{-0.5x}\left(2x - \frac{1}{2}x^2\right)$$

There is an absolute maximum of 1.356 at about 0.6085 and an absolute minimum of 0.5 at -1.

27. $f(x) = x + e^{-3x}$; [-1, 3]

$$f'(x) = 1 - 3e^{-3x}$$

There is an absolute maximum at $x = 3$. There is no absolute minimum, as can be seen by looking at the graph of $f$.

28. $f(x) = \frac{x^3 + 2x + 5}{x^4 + 3x^2 + 10}$; [-3, 0]

The indicated domain tells us the $x$-values to use for the viewing window, but we must experiment to find a suitable range for the $y$-values. In order to show the absolute extrema on $[-3, 0]$, we find that a suitable window is $[-3, 0]$ by $[0, 1.5]$ with Xsc1 = 0.1, Ysc1 = 0.1.

From the graph, we see that on $[-3, 0]$, $f$ has an absolute maximum of 0.5 at 0 and an absolute minimum of 8.10 at about -2.35.

29. $f(x) = \frac{-5x^4 + 2x^3 + 3x^2 + 9}{x^4 - x^3 + 2x^2 + 7}$; [-1, 1]

The indicated domain tells us the $x$-values to use for the viewing window, but we must experiment to find a suitable range for the $y$-values. In order to show the absolute extrema on $[-1, 1]$, we find that a suitable window is $[-1, 1]$ by $[0, 1.5]$ with Xsc1 = 0.1, Ysc1 = 0.1.

From the graph, we see that on $[-1, 1]$, $f$ has an absolute maximum of 1.356 at about 0.6085 and an absolute minimum of 0.5 at -1.

30. $f(x) = 12 - x - \frac{9}{x}$, $x > 0$

$$f'(x) = -1 + \frac{9}{x^2} = \frac{(3 + x)(3 - x)}{x^2}$$

There is an absolute maximum at $x = 3$. There is no absolute minimum, as can be seen by looking at the graph of $f$.

31. $f(x) = 2x + \frac{8}{x^2} + 1$, $x > 0$

$$f'(x) = 2 - \frac{16}{x^3} = \frac{2x^3 - 16}{x^3} = \frac{2(x - 2)(x^2 + 2x + 4)}{x^3}$$

Since the specified domain is $(0, \infty)$, a critical number is $x = 2$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>7.00</td>
</tr>
</tbody>
</table>

There is an absolute minimum at $x = 2$; there is no absolute maximum, as can be seen by looking at the graph of $f$. 
32. \( f(x) = x^4 - 4x^3 + 4x^2 + 1 \)
\[
f'(x) = 4x^3 - 12x^2 + 8x
\]
\[
= 4x(x^2 - 3x + 2)
\]
\[
= 4x(x - 2)(x - 1)
\]
The critical numbers are 0, 1, and 2.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

There is no absolute maximum, as can be seen by looking at the graph of \( f \). There is an absolute minimum at \( x = 0 \) and \( x = 2 \).

33. \( f(x) = -3x^4 + 8x^3 + 18x^2 + 2 \)
\[
f'(x) = -12x^3 + 24x^2 + 36x
\]
\[
= -12x(x^2 - 2x - 3)
\]
\[
= -12x(x - 3)(x + 1)
\]

Critical numbers are 0, 3, and \(-1\).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-1)</td>
<td>9</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>137</td>
</tr>
</tbody>
</table>

There is an absolute maximum at \( x = 3 \); there is no absolute minimum, as can be seen by looking at the graph of \( f \).

34. \( f(x) = \frac{x}{x^2 + 1} \)
\[
f'(x) = \frac{(x^2 + 1) - x(2x)}{(x^2 + 1)^2}
\]
\[
= \frac{x^2 + 1 - 2x^2}{(x^2 + 1)^2}
\]
\[
= \frac{1 - x^2}{(x^2 + 1)^2}
\]
\[
= \frac{(1 + x)(1 - x)}{(x^2 + 1)^2}
\]
The critical numbers are \(-1\) and 1.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-1)</td>
<td>(-0.5)</td>
</tr>
<tr>
<td>1</td>
<td>0.5</td>
</tr>
</tbody>
</table>

There is an absolute maximum of 0.5 at \( x = 1 \) and an absolute minimum of \(-0.5\) at \( x = -1 \). This can be verified by looking at the graph of \( f \).

35. \( f(x) = \frac{x - 1}{x^2 + 2x + 6} \)
\[
f'(x) = \frac{(x^2 + 2x + 6)(1) - (x - 1)(2x + 2)}{(x^2 + 2x + 6)^2}
\]
\[
= \frac{x^2 + 2x + 6 - 2x^2 - 2x}{(x^2 + 2x + 6)^2}
\]
\[
= \frac{-x^2 + 2x + 8}{(x^2 + 2x + 6)^2}
\]
\[
= \frac{-x^2 - 2x - 8}{(x^2 + 2x + 6)^2}
\]
\[
= \frac{-x - 4)(x + 2)}{(x^2 + 2x + 6)^2}
\]

Critical numbers are 4 and \(-2\).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-2)</td>
<td>(-0.5)</td>
</tr>
<tr>
<td>4</td>
<td>0.1</td>
</tr>
</tbody>
</table>

There is an absolute maximum at \( x = 4 \) and an absolute minimum at \( x = -2 \). This can be verified by looking at the graph of \( f \).

36. \( f(x) = x \ln x \)
\[
f'(x) = x \cdot \frac{1}{x} + 1 \cdot \ln x
\]
\[
= 1 + \ln x
\]
\[
f'(x) = 0 \text{ when } x = e^{-1}, \text{ and } f'(x) \text{ does not exist when } x \leq 0. \text{ The only critical number is } e^{-1}.
\]
\[
\begin{array}{c|c|c|c|c|c}
\hline
\( x \) & \( e^{-1} \) & \( f(x) \) \\
\hline
\hline
\end{array}
\]
\[
= e^{-1} \approx -0.3679
\]

There is an absolute minimum of \(-0.3679\) at \( x = e^{-1} \). There is no absolute maximum, as can be seen by looking at the graph of \( f \).

37. \( f(x) = \frac{\ln x}{x^3} \)
\[
f'(x) = \frac{x^3 \cdot \frac{1}{x} - 3x^2 \ln x}{x^6}
\]
\[
= \frac{x^2 - 3x^2 \ln x}{x^6}
\]
\[
= \frac{x^2(1 - 3 \ln x)}{x^6}
\]
\[
= \frac{1 - 3 \ln x}{x^4}
\]
Section 6.1 Absolute Extrema

38. $f(x) = 2x - 3x^{2/3}$

Let $f(x) = 2x - 3x^{2/3}$ and $f'(x)$ does not exist when $x \leq 0$. The only critical number is $e^{1/3}$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e^{1/3}$</td>
<td>$\frac{1}{3} e^{-1} \approx 0.1226$</td>
</tr>
</tbody>
</table>

There is an absolute maximum of 0.1226 at $x = e^{1/3}$. There is no absolute minimum, as can be seen by looking at the graph of $f$.

By looking at the graph, there are relative maxima of 313 in 1997 and an absolute minimum of 134 in 2004. There are relative minima of 6599 in 1999 and 7556 in 2003.

(a) On $[-1, 0.5]$:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>-5</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>0.5</td>
<td>-0.88988</td>
</tr>
</tbody>
</table>

Absolute minimum of -5 at $x = -1$; absolute maximum of 0 at $x = 0$.

(b) On $[0.5, 2]$:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>-0.88988</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>2</td>
<td>-0.7622</td>
</tr>
</tbody>
</table>

Absolute maximum of about -0.76 at $x = 2$; absolute minimum of -1 at $x = 1$.

39. Let $P(x)$ be the perimeter of the rectangle with vertices $(0, 0), (x, 0), (x, f(x))$, and $(0, f(x))$ for $x > 0$ when $f(x) = e^{-2x}$.

The length of the rectangle is $x$ and the width is given by $e^{-2x}$. Therefore, an equation for the perimeter is

$$P(x) = x + e^{-2x} + x + e^{-2x} = 2(x + e^{-2x}).$$

There is an absolute minimum of 1.693 at $x = \frac{\ln 2}{2}$. There is no absolute maximum, as can be seen by looking at the graph of $P$. Therefore, the correct statement is a.

40. (a) By looking at the graph, there are relative maxima of 8046 in 1996, 8496 in 2001, and 7556 in 2004. There are relative minima of 6599 in 1999 and 7465 in 2003.

(b) Annual bank robberies reached an absolute maximum of 8496 in 2001 and an absolute minimum of 6599 in 1999.


(b) Annual bank burglaries reached an absolute maximum of 413 in 1997 and an absolute minimum of 131 in 2003.

42. $P(x) = -0.02x^3 + 600x - 20,000$, [50, 150]

<table>
<thead>
<tr>
<th>$x$</th>
<th>$P(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>7500</td>
</tr>
<tr>
<td>100</td>
<td>20,000</td>
</tr>
<tr>
<td>150</td>
<td>2500</td>
</tr>
</tbody>
</table>

There are critical numbers at $x = -100$ and $x = 100$. Only $x = 100$ is in the domain of $P(x)$. The function is defined at $x = 100$.
Chapter 6  APPLICATIONS OF THE DERIVATIVE

The maximum profit is $20,000, which occurs when 100 units per week are made.

43. \( P(x) = -x^3 + 9x^2 + 120x - 400, \ x \geq 5 \)

\[
P'(x) = -3x^2 + 18x + 120
= -3(x^2 - 6x - 40)
= -3(x - 10)(x + 4) = 0
\]

\( x = 10 \) or \( x = -4 \)

\(-4\) is not relevant since \( x \geq 5 \), so the only critical number is 10.

The graph of \( P'(x) \) is a parabola that opens downward, so \( P'(x) > 0 \) on the interval \([5, 10)\) and \( P'(x) < 0 \) on the interval \((10, \infty)\). Thus, \( P(x) \) is a maximum at \( x = 10 \).

Since \( x \) is measured in hundred thousands, 10 hundred thousand or 1,000,000 tires must be sold to maximize profit.

Also,

\[
P(10) = -(10)^3 + 9(10)^2 + 120(10) - 400
= 700.
\]

The maximum profit is $700 thousand or $700,000.

44. \( C(x) = 81x^2 + 17x + 324 \)

(a) \( 1 \leq x \leq 10 \)

\[
\overline{C}(x) = \frac{C(x)}{x} = \frac{81x^2 + 17x + 324}{x}
= 81x + 17 + \frac{324}{x}
\]

\[
\overline{C}'(x) = 81 - \frac{324}{x^2} = 0 \text{ when } x^2 = \frac{324}{81}
= 4
\]

\( x = \pm 2 \).

\(-2\) is not meaningful in this application and is not in the given domain.

Test 2 for minimum.

\[
\overline{C}'(1) = -243 < 0
\overline{C}'(3) = 45 > 0
\]

\( \overline{C}(x) \) is a minimum when \( x = 2 \).

\[
\overline{C}(2) = 81(2) + 17 + \frac{324}{2} = 341
\]

The minimum on the interval \( 1 \leq x \leq 10 \) is 341.

(b) \( 10 \leq x \leq 20 \)

There are no critical numbers in this interval. Test the endpoints.

\[
\overline{C}(10) = 859.4
\overline{C}(20) = 1620 + 17 + 16.2
= 1633.2
\]

The minimum on the interval \( 10 \leq x \leq 20 \) is 859.4.

45. \( C(x) = x^3 + 37x + 250 \)

(a) \( 1 \leq x \leq 10 \)

\[
\overline{C}(x) = \frac{C(x)}{x} = \frac{x^3 + 37x + 250}{x}
= x^2 + 37 + \frac{250}{x}
\]

\[
\overline{C}'(x) = 2x - \frac{250}{x^2} = 0 \text{ when } x^2 = \frac{250}{2}
= 125
\]

\( x = 5 \).

Test for relative minimum.

\[
\overline{C}'(4) = -7.625 < 0
\overline{C}'(6) = 5.0556 > 0
\overline{C}'(5) = 112
\overline{C}'(1) = 1 + 37 + 250 = 288
\overline{C}'(10) = 100 + 37 + 25 = 162
\]

The minimum on the interval \( 1 \leq x \leq 10 \) is 112.

(b) \( 10 \leq x \leq 20 \)

There are no critical values in this interval. Check the endpoints.

\[
\overline{C}(10) = 162
\overline{C}(20) = 400 + 37 + 12.5 = 449.5
\]

The minimum on the interval \( 10 \leq x \leq 20 \) is 162.

46. The value \( x = 20 \) minimizes \( \frac{f(x)}{x} \) because this is the point where the line from the origin to the curve is tangent to the curve.

A production level of 20 units results in the minimum cost per unit.

47. The value \( x = 11 \) minimizes \( \frac{f(x)}{x} \) because this is the point where the line from the origin to the curve is tangent to the curve.

A production level of 11 units results in the minimum cost per unit.
48. The value \( x = 300 \) maximizes \( \frac{f(x)}{x} \) because this is the point where the line from the origin to the curve is tangent to the curve.
A production level of 300 units results in the maximum profit per item produced.

49. The value \( x = 100 \) maximizes \( \frac{f(x)}{x} \) because this is the point where the line from the origin to the curve is tangent to the curve.
A production level of 100 units results in the maximum profit per item produced.

50. \( f(x) = \frac{x^2 + 36}{2x}, \ 1 \leq x \leq 12 \)

\[
f'(x) = \frac{2x(2x) - (x^2 + 36)(2)}{(2x)^2} = \frac{4x^2 - 2x^2 - 72}{4x^2} = \frac{2x^2 - 72}{4x^2} = \frac{2(x^2 - 36)}{4x^2} = \frac{(x + 6)(x - 6)}{2x^2}
\]

\( f'(x) = 0 \) when \( x = 6 \) and when \( x = -6 \). Only 6 is in the interval \( 1 \leq x \leq 12 \).
Test for relative maximum or minimum.

\[
f''(5) = \frac{(11)(-1)}{50} < 0
\]

\[
f''(7) = \frac{(13)(1)}{98} > 0
\]

The minimum occurs at \( x = 6 \), or at 6 months.
Since \( f(6) = 6 \), \( f(1) = 18.5 \), and \( f(12) = 7.5 \), the minimum percent is 6%.

51. \( S(x) = -x^3 + 3x^2 + 360x + 5000; \ 6 \leq x \leq 20 \)

\[
S'(x) = -3x^2 + 6x + 360 = -3(x^2 - 2x - 120)
\]

\[
S'(x) = -3(x - 12)(x + 10) = 0
\]

\( x = 12 \) or \( x = -10 \) (not in the interval)

\[
\begin{array}{c|c}
 x & f(x) \\
 6 & 7052 \\
 12 & 8024 \\
 10 & 7900 \\
\end{array}
\]

12\(^\circ\) is the temperature that produces the maximum number of salmon.

52. Since we are only interested in the length during weeks 22 through 28, the domain of the function for this problem is \([22, 28]\). We now look for any critical numbers in this interval. We find

\[
L'(t) = 0.788 - 0.02t
\]

There is a critical number at \( t = \frac{0.788}{0.02} = 39.4 \), which is not in the interval. Thus, the maximum value will occur at one of the endpoints.

\[
\begin{array}{c|c}
 t & L(t) \\
 22 & 5.4 \\
 28 & 7.2 \\
\end{array}
\]

The maximum length is about 7.2 millimeters.

53. The function is defined on the interval \([15, 46]\). We look first for critical numbers in the interval. We find

\[
R'(T) = -0.00021T^2 + 0.0802T - 1.6572
\]

Using our graphing calculator, we find one critical number in the interval at about 21.92

\[
\begin{array}{c|c}
 T & R(T) \\
 15 & 81.01 \\
 21.92 & 79.29 \\
 46 & 98.89 \\
\end{array}
\]

The relative humidity is minimized at about 21.92\(^\circ\)C.

54. \( M(x) = -\frac{1}{45}x^2 + 2x - 20; \ 30 \leq x \leq 65 \)

\[
M'(x) = -\frac{1}{45}(2x) + 2 = -\frac{2x}{45} + 2
\]

When \( M'(x) = 0, \)

\[
-\frac{2x}{45} + 2 = 0
\]

\[
2 = \frac{2x}{45}
\]

\[
x = 45
\]

\[
\frac{45}{9} \approx 16.1
\]

The absolute maximum miles per gallon is 25 and the absolute minimum miles per gallon is about 16.1.
55. \( M(x) = -0.015x^2 + 1.31x - 7.3, 30 \leq x \leq 60 \)

\[ M'(x) = -0.03x + 1.31 = 0 \]

\[ x \approx 43.7 \]

\[
\begin{array}{c|c}
 x & M(x) \\
\hline
30 & 18.5 \\
43.7 & 21.30 \\
60 & 17.3 \\
\end{array}
\]

The absolute maximum of 21.30 mpg occurs at 43.7 mph. The absolute minimum of 17.3 mpg occurs at 60 mph.

56. Total area \( A(x) = \pi \left( \frac{x}{2\pi} \right)^2 + \left( \frac{12-x}{4} \right)^2 \)

\[ = \frac{x^2}{4\pi^2} + \frac{(12-x)^2}{16} \]

\[ A'(x) = \frac{x}{2\pi} - \frac{12-x}{8} = 0 \]

\[ \frac{4x - \pi(12-x)}{8\pi} = 0 \]

\[ x = \frac{12\pi}{4+\pi} \approx 5.28 \]

\[
\begin{array}{c|c}
 x & \text{Area} \\
\hline
0 & 9 \\
5.28 & 5.04 \\
12 & 11.46 \\
\end{array}
\]

The total area is minimized when the piece used to form the circle is \( \frac{12\pi}{4+\pi} \) feet, or about 5.28 feet long.

57. Total area \( A(x) \)

\[ = \pi \left( \frac{x}{2\pi} \right)^2 + \left( \frac{12-x}{4} \right)^2 \]

\[ = \frac{x^2}{4\pi^2} + \frac{(12-x)^2}{16} \]

\[ A'(x) = \frac{x}{2\pi} - \frac{12-x}{8} = 0 \]

\[ \frac{4x - \pi(12-x)}{8\pi} = 0 \]

\[ x = \frac{12\pi}{4+\pi} \approx 5.28 \]

\[
\begin{array}{c|c}
 x & \text{Area} \\
\hline
0 & 9 \\
5.28 & 5.04 \\
12 & 11.46 \\
\end{array}
\]

The total area is maximized when all 12 feet of wire are used to form the circle.

58. For the solution to Exercise 56, the piece of length \( x \) used to form the circle is \( \frac{12\pi}{4+\pi} \) feet. The circle can be inscribed inside the square if the side of the square equals the diameter of the circle (that is, twice the radius).

\[
\text{side of the square} = 2(\text{radius})
\]

\[ \frac{12-x}{4} = 2 \left( \frac{x}{2\pi} \right) \]

\[ \frac{12-x}{4} = \frac{x}{\pi} \]

\[ 4x = 12\pi - \pi x \]

\[ x(4+\pi) = 12\pi \]

\[ x = \frac{12\pi}{4+\pi} \]

Therefore, the circle formed by piece of length \( x = \frac{12\pi}{4+\pi} \) can be inscribed inside the square.

59. (a) \( I(p) = -p \ln p - (1-p) \ln (1-p) \)

\[ I'(p) = -p \left( \frac{1}{p} \right) + \ln p (-1) \]

\[ - \left[ (1-p) \left( \frac{-1}{1-p} \right) + \ln (1-p) [-1] \right] \]

\[ = -1 - \ln p + 1 + \ln (1-p) \]

\[ = -\ln p + \ln (1-p) \]

(b) \(-\ln p + \ln (1-p) = 0 \)

\[ \ln (1-p) = \ln p \]

\[ 1 - p = p \]

\[ 1 = 2p \]

\[ \frac{1}{2} = p \]

\[ I'(0.25) = 1.0986 \]

\[ I'(0.75) = -1.099 \]

There is a relative maximum of 0.693 at \( p = \frac{1}{2} \).
6.2 Applications of Extrema

1. \( x + y = 180, \ P = xy \)

   (a) \( y = 180 - x \)

   (b) \( P = xy = x(180 - x) \)

   (c) Since \( y = 180 - x \) and \( x \) and \( y \) are nonnegative numbers, \( x \geq 0 \) and \( 180 - x \geq 0 \) or \( x \leq 180 \). The domain of \( P \) is \([0, 180]\).

   (d) \( P'(x) = 180 - 2x \)

\[
\begin{array}{c|c}
  x & P \\
  \hline
  0 & 0 \\
  90 & 8100 \\
  180 & 0 \\
\end{array}
\]

   (f) From the chart, the maximum value of \( P \) is 8100; this occurs when \( x = 90 \) and \( y = 90 \).

2. \( x + y = 140 \)

Minimize \( x^2 + y^2 \).

   (a) \( y = 140 - x \)

   (b) Let \( P = x^2 + y^2 = x^2 + (140 - x)^2 \)

\[
= x^2 + 19,600 - 280x + x^2 \\
= 2x^2 - 280x + 19,600.
\]

   (c) Since \( y = 140 - x \) and \( x \) and \( y \) are nonnegative numbers, the domain of \( P \) is \([0, 140]\).

   (d) \( P' = 4x - 280 \)

\[
\begin{array}{c|c}
  x & P \\
  \hline
  0 & 19,600 \\
  70 & 9800 \\
  140 & 19,600 \\
\end{array}
\]

   (f) The minimum value of \( x^2 + y^2 \) occurs when \( x = 70 \) and \( y = 140 - x = 140 - 70 = 70 \). The minimum value is 9800.

3. \( x + y = 90 \)

Minimize \( x^2y \).

   (a) \( y = 90 - x \)

   (b) Let \( P = x^2y = x^2(90 - x) \)

\[
= 90x^2 - x^3.
\]

   (c) Since \( y = 90 - x \) and \( x \) and \( y \) are nonnegative numbers, the domain of \( P \) is \([0, 90]\).

   (d) \( P' = 180x - 3x^2 \)

\[
180x - 3x^2 = 0 \\
3x(60 - x) = 0 \\
x = 0 \text{ or } x = 60
\]

   (e) \[
\begin{array}{c|c}
  x & P \\
  \hline
  0 & 0 \\
  60 & 108,000 \\
  90 & 0 \\
\end{array}
\]

   (f) The maximum value of \( x^2y \) occurs when \( x = 60 \) and \( y = 30 \). The maximum value is 108,000.

4. \( x + y = 105 \)

Maximize \( xy^2 \).

   (a) \( y = 105 - x \)

   (b) Let \( P = xy^2 = x(105 - x)^2 \)

\[
= x(11,025 - 210x + x^2) \\
= 11,025x - 210x^2 + x^3.
\]

   (c) Since \( y = 105 - x \) and \( x \) and \( y \) are nonnegative numbers, the domain of \( P \) is \([0, 105]\).

   (d) \( P' = 11,025 - 420x + 3x^2 \)

\[
3x^2 - 420x + 11,025 = 0 \\
3(x^2 - 140x + 3675) = 0 \\
3(x - 35)(x - 105) = 0 \\
x = 35 \text{ or } x = 105
\]

   (e) \[
\begin{array}{c|c}
  x & P \\
  \hline
  0 & 0 \\
  35 & 171,500 \\
  105 & 0 \\
\end{array}
\]

   (f) The maximum value of \( xy^2 \) occurs when \( x = 35 \) and \( y = 70 \). The maximum value is 171,500.
5. \(C(x) = \frac{1}{2}x^3 + 2x^2 - 3x + 35\)

The average cost function is

\[A(x) = \frac{C(x)}{x} = \frac{\frac{1}{2}x^3 + 2x^2 - 3x + 35}{x} = \frac{1}{2}x^2 + 2x - 3 + \frac{35}{x}\]

or \(\frac{1}{2}x^2 + 2x - 3 + 35x^{-1}\).

Then

\[A'(x) = x + 2 - 35x^{-2}\]

or \(x + 2 - \frac{35}{x^2}\).

Graph \(y = A'(x)\) on a graphing calculator. A suitable choice for the viewing window is \([0, 10]\) by \([-10, 10]\). (Negative values of \(x\) are not meaningful in this application.) Using the calculator, we see that the graph has an \(x\)-intercept or “zero” at \(x \approx 2.722\). Thus, 2.722 is a critical number.

Now graph \(y = A(x)\) and use this graph to confirm that a minimum occurs at \(x \approx 2.722\).

Thus, the average cost is smallest at \(x \approx 2.722\).

6. \(C(x) = 10 + 20x^{1/2} + 16x^{3/2}\)

The average cost function is

\[A(x) = \frac{C(x)}{x} = \frac{10 + 20x^{1/2} + 16x^{3/2}}{x} = \frac{10}{x} + 20x^{-1/2} + 16x^{1/2}\]

or \(10x^{-1} + 20x^{-1/2} + 16x^{1/2}\).

Then

\[A'(x) = -10x^{-2} - 10x^{-3/2} + 8x^{-1/2}\]

Graph \(y = A'(x)\) on a graphing calculator. A suitable choice for the viewing window is \([0, 10]\) by \([-10, 10]\). (Negative values of \(x\) are not meaningful in this application.) We see that this graph has one \(x\)-intercept or “zero.” Using the calculator, we find that this \(x\)-value is about 2.110, which shows that 2.110 is the only critical number of \(A\).

Now graph \(y = A(x)\) and use this graph to confirm that a minimum occurs at \(x \approx 2.110\). Thus, the average cost is smallest at \(x \approx 2.110\).

7. \(p(x) = 160 - x\frac{x}{10}\)

(a) Revenue from sale of \(x\) thousand candy bars:

\[R(x) = 1000xp = 1000x \left(160 - \frac{x}{10}\right) = 160,000x - 100x^2\]

(b) \(R'(x) = 160,000 - 200x\)

\[160,000 - 200x = 0\]

\[160,000 = 200x\]

\[800 = x\]

The maximum revenue occurs when 800 thousand bars are sold.

(c) \(R(800) = 160,000(800) - 100(800)^2 = 64,000,000\)

The maximum revenue is 64,000,000 cents.

8. \(p(x) = 12 - \frac{x}{8}\)

(a) Revenue from \(x\) thousand compact discs:

\[R(x) = 1000xp = 1000x \left(12 - \frac{x}{8}\right) = 12,000x - 125x^2\]

(b) \(R'(x) = 12,000 - 250x\)

\[12,000 - 250x = 0\]

\[12,000 = 250x\]

\[48 = x\]

The maximum revenue occurs when 48 thousand compact discs are sold.

(c) \(R(48) = 12,000(48) - 125(48)^2 = 288,000\)

The maximum revenue is $288,000.

9. Let \(x = \) the width and \(y = \) the length.

(a) The perimeter is

\[P = 2x + y = 1400, \quad \text{so} \quad y = 1400 - 2x.\]
(b) Area = \( xy = x(1400 - 2x) \)
\[ A(x) = 1400x - 2x^2 \]

(c) \[ A' = 1400 - 4x \]
\[ 1400 - 4x = 0 \]
\[ 1400 = 4x \]
\[ 350 = x \]

\( A'' = -4 \), which implies that \( x = 350 \) m leads to the maximum area.

(d) If \( x = 350 \),
\[ y = 1400 - 2(350) = 700. \]
The maximum area is \( (350)(700) = 245,000 \text{ m}^2 \).

10. Let \( x \) = length of field
\( y \) = width of field.
Perimeter:
\[ P = 2x + 2y = 300 \]
\[ x + y = 150 \]
\[ y = 150 - x \]

Area:
\[ A = xy = x(150 - x) = 150x - x^2 \]
Thus,
\[ A(x) = 150x - x^2 \]
\[ A'(x) = 150 - 2x. \]

\( A'(x) = 0 \) when
\[ 150 - 2x = 0 \]
\[ x = 75. \]

\( A''(x) = -2 \), so \( A''(75) = -2 < 0 \), which confirms that a maximum value occurs at \( x = 75 \).
If \( x = 75 \),
\[ y = 150 - x = 150 - 75 = 75. \]

A maximum area occurs when the length is 75 m and the width is 75 m.

11. Let \( x \) = the width of the rectangle
\( y \) = the total length of the rectangle.

An equation for the fencing is
\[ 3600 = 4x + 2y \]
\[ 2y = 3600 - 4x \]
\[ y = 1800 - 2x. \]

Area = \( xy = x(1800 - 2x) \)
\[ A(x) = 1800x - 2x^2 \]
\[ A' = 1800 - 4x \]
\[ 1800 - 4x = 0 \]
\[ 1800 = 4x \]
\[ 450 = x \]

\( A'' = -4 \), which implies that \( x = 450 \) is the location of a maximum.
If \( x = 450 \), \( y = 1800 - 2(450) = 900. \)
The maximum area is \( (450)(900) = 405,000 \text{ m}^2 \).

12. Let \( x \) = the length at $2.50 per foot
\( y \) = the width at $3.20 per foot.

\[ xy = 20,000 \]
\[ y = \frac{20,000}{x} \]

Perimeter = \( 2x + 2y = 2x + \frac{40,000}{x} \)

Cost = \( C(x) = 2x(2.5) + \frac{40,000}{x}(3.2) = 5x + \frac{128,000}{x} \)

Minimize cost:
\[ C'(x) = 5 - \frac{128,000}{x^2} \]
\[ 5 - \frac{128,000}{x^2} = 0 \]
\[ 5 = \frac{128,000}{x^2} \]
\[ 5x^2 = 128,000 \]
\[ x^2 = 25,600 \]
\[ x = 160 \]
\[ y = \frac{20,000}{160} = 125 \]

320 ft at $2.50 per foot will cost $800. 250 ft at $3.20 per foot will cost $800. The entire cost will be $1600.
13. Let \( x \) = length at $1.50 per meter \\
\( y \) = width at $3 per meter. \\
\[
xy = 25,600 \\
y = \frac{25,600}{x}
\]
Perimeter \( = x + 2y = x + \frac{51,200}{x} \)
Cost \( = C(x) = x(1.5) + \frac{51,200}{x} (3) \)
\[
= 1.5x + \frac{153,600}{x}
\]
Minimize cost:
\[
C'(x) = 1.5 - \frac{153,600}{x^2} = 0 \\
1.5 = \frac{153,600}{x^2} \\
1.5x^2 = 153,600 \\
x^2 = 102,400 \\
x = 320 \\
y = \frac{25,600}{320} = 80
\]
320 m at $1.50 per meter will cost $480. 160 m at $3 per meter will cost $480. The total cost will be $960.

14. Let \( x \) = the number of seats. 
Profit is 6 dollars per seat for \( 0 \leq x \leq 50 \). 
Profit (in dollars) is \( 6 - 0.10(x - 50) \) per seat for \( x > 50 \). 
We expect that the number of seats which makes the total profit a maximum will be greater than 50 because after 50 the profit is still increasing, though at a slower rate. (Thus we know the function is concave down and its one extremum will be a maximum.)

(a) The total profit for \( x \) seats is 
\[
P(x) = [6 - 0.1(x - 50)]x \\
= (6 - 0.1x + 5)x \\
= (11 - 0.1x)x \\
= 11x - 0.1x^2.
\]
\[
P'(x) = 11 - 0.2x \\
11 - 0.2x = 0 \\
11 = 0.2x \\
x = 55
\]
55 seats will produce maximum profit.

(b) \( P(55) = 11(55) - 0.10(55^2) \)
\[
= 605 - 0.10(3025) \\
= 302.5
\]
The maximum profit is $302.50.

15. Let \( x \) = the number of days to wait. 
\[
\frac{12,000}{100} = 120 = \text{the number of 100-lb groups collected already.}
\]
Then \( 7.5 - 0.15x \) = the price per 100 lb; 
\( 4x \) = the number of 100-lb groups collected per day; 
\( 120 + 4x \) = total number of 100-lb groups collected.

\[
\text{Revenue} = R(x) \\
= (7.5 - 0.15x)(120 + 4x) \\
= 900 + 12x - 0.6x^2
\]
\[
R'(x) = 12 - 1.2x = 0 \\
x = 10
\]
\[
R''(x) = -1.2 < 0 \text{ so } R(x) \text{ is maximized at } x = 10.
\]
The scouts should wait 10 days at which time their income will be maximized at 
\[
R(10) = 900 + 12(10) - 0.6(10)^2 = $960.
\]

16. Let \( x \) = a side of the base \\
\( h \) = the height of the box.

An equation for the volume of the box is 
\[
V = x^2h, \\
\text{so } 32 = x^2h \\
h = \frac{32}{x^2}
\]
The box is open at the top so the area of the surface material \( m(x) \) in square inches is the area of the base plus the area of the four sides.
Section 6.2  Applications of Extrema

389

\[ m(x) = x^2 + 4xh \]
\[ m(x) = x^2 + 4x \left( \frac{32}{x^2} \right) \]
\[ m(x) = x^2 + \frac{128}{x} \]
\[ m'(x) = 2x - \frac{128}{x^2} \]
\[ \frac{2x^3 - 128}{x^2} = 0 \]
\[ 2x^3 - 128 = 0 \]
\[ 2(x^3 - 64) = 0 \]
\[ x = 4 \]

\[ m'(x) = 2 + \frac{256}{x^3} > 0 \text{ since } x > 0. \]
So, \( x = 4 \) minimizes the surface material.

If \( x = 4 \),
\[ h = \frac{32}{x^2} = \frac{32}{16} = 2. \]

The dimensions that will minimize the surface material are \( 4 \text{ in by } 4 \text{ in by } 2 \text{ in} \).

17. Let \( x \) = the number of refunds.
Then \( 535 - 5x \) = the cost per passenger
and \( 85 + x \) = the number of passengers.

(a) Revenue \( = R(x) = (535 - 5x)(85 + x) \)
\[ R'(x) = 110 - 10x = 0 \]
\[ x = 11 \]

\[ R''(x) = -10 < 0, \text{ so } R(x) \text{ is maximized when } \]
\[ x = 11. \]
Thus, the number of passengers that will maximize revenue is \( 85 + 11 = 96. \)

(b) \[ R(11) = 45,475 + 110(11) - 5(11)^2 \]
\[ = 46,080 \]
The maximum revenue is \( $46,080. \)

18. Let \( x \) = the width.
Then \( 2x \) = the length
and \( h \) = the height.

An equation for volume is
\[ V = (2x)(x)h = 2x^2h \]
\[ 36 = 2x^2h. \]
So, \( h = \frac{18}{x}. \)

The surface area \( S(x) \) is the sum of the areas of the base and the four sides.
\[ S(x) = (2x)(x) + 2xh + 2(2x)h \]
\[ = 2x^2 + 6xh \]
\[ = 2x^2 + 6x \left( \frac{18}{x^2} \right) \]
\[ = 2x^2 + \frac{108}{x} \]
\[ S'(x) = 4x - \frac{108}{x^2} \]
\[ \frac{4x^3 - 108}{x^2} = 0 \]
\[ 4(x^3 - 27) = 0 \]
\[ x = 3 \]

\[ S''(x) = 4 + \frac{108(2)}{x^3} \]
\[ = 4 + \frac{216}{x^3} > 0 \text{ since } x > 0. \]
So \( x = 3 \) minimizes the surface material.
If \( x = 3 \),
\[ h = \frac{18}{x^2} = \frac{18}{9} = 2. \]
The dimensions are \( 3 \text{ ft by } 6 \text{ ft by } 2 \text{ ft}. \)

19. Let \( x \) = the length of a side of the top and bottom.
Then \( x^2 \) = the area of the top and bottom
and \( (3)(2x^2) = \) the cost for the top and bottom.

Let \( y \) = depth of box.
Then \( xy \) = the area of one side,
\( 4xy \) = the total area of the sides,
and \( (1.50)(4xy) = \) the cost of the sides.
The total cost is
\[ C(x) = (3)(2x^2) + (1.50)(4xy) = 6x^2 + 6xy. \]
The volume is
\[ V = 16,000 = x^2y. \]
\[ y = \frac{16,000}{x^2} \]
\[ C(x) = 6x^2 + 6x \left( \frac{16,000}{x^2} \right) = 6x^2 + \frac{96,000}{x} \]
\[ C'(x) = 12x - \frac{96,000}{x^3} = 0 \]
\[ x = 20 \]
Chapter 6 APPLICATIONS OF THE DERIVATIVE

20. 120 centimeters of ribbon are available; it will cover 4 heights and 8 radii.

\[4h + 8r = 120\]
\[h + 2r = 30\]
\[h = 30 - 2r\]

\[V = \pi r^2h\]
\[V = \pi r^2(30 - 2r)\]
\[= 30\pi r^2 - 2\pi r^3\]

Maximize volume.

\[V' = 60\pi r - 6\pi r^2\]
\[60\pi r - 6\pi r^2 = 0\]
\[6\pi r(10 - r) = 0\]
\[r = 0 \quad \text{or} \quad r = 10\]

If \(r = 0\), there is no box, so we discard this value. \(V'' = 60\pi - 12\pi r < 0\) for \(r = 10\), which implies that \(r = 10\) gives maximum volume.

When \(r = 10\), \(h = 30 - 2(10) = 10\).

The volume is maximized when the radius and height are both 10 cm.

21. (a) \(S = 2\pi r^2 + 2\pi rh\), \(V = \pi r^2h\)

\[S = 2\pi r^2 + \frac{2V}{r}\]

Treat \(V\) as a constant.

\[S' = 4\pi r - \frac{2V}{r^2}\]
\[4\pi r - \frac{2V}{r^2} = 0\]
\[4\pi r^3 - 2V = 0\]
\[r = \frac{\sqrt[3]{2V}}{\pi}\]
\[2r = h\]

22. \(V = \pi r^2h = 16\)

\[h = \frac{16}{\pi r^2}\]

The total cost is the sum of the cost of the top and bottom and the cost of the sides.

\[C = 2(2)(\pi r^2) + 1(2\pi r)\left(\frac{16}{\pi r^2}\right)\]
\[= 4\pi r^2 + \frac{32}{r}\]

Minimize cost.

\[C' = 8\pi r - \frac{32}{r^2}\]
\[8\pi r - \frac{32}{r^2} = 0\]
\[8\pi r^3 = 32\]
\[\pi r^3 = 4\]
\[r = \sqrt[3]{\frac{4}{\pi}} \approx 1.08\]

\[h = \frac{16}{\pi (1.08)^2} \approx 4.34\]

The radius should be 1.08 ft and the height should be 4.34 ft. If these rounded values for the height and radius are used, the cost is

\[\$2(2)(\pi (1.08)^2) + \$1(2\pi r(4.34))\]
\[= 4\pi (1.08)^2 + 2\pi (1.08)(4.34)\]
\[= \$44.11.\]

23. Let \(x\) = the length of the side of the cutout square.

Then \(3 - 2x\) = the width of the box and \(8 - 2x\) = the length of the box.

\[V(x) = x(3 - 2x)(8 - 2x)\]
\[= 4x^3 - 22x^2 + 24x\]

The domain of \(V\) is \((0, \frac{3}{2})\).

Maximize the volume.

\[V'(x) = 12x^2 - 44x + 24\]
\[12x^2 - 44x + 24 = 0\]
\[4(3x^2 - 11x + 6) = 0\]
\[4(3x - 2)(x - 3) = 0\]
\[x = \frac{2}{3} \quad \text{or} \quad x = 3\]
Section 6.2 Applications of Extrema

3 is not in the domain of \( V \).

\[
V''(x) = 24x - 44
\]

\[
V'' \left( \frac{2}{3} \right) = -28 < 0
\]

This implies that \( V \) is maximized when \( x = \frac{2}{3} \).
The box will have maximum volume when \( x = \frac{2}{3} \) ft or 8 in.

24. (a) From Example 3, the area of the base is \((12 - 2x)(12 - 2x) = 4x^2 - 48x + 144\) and the total area of all four walls is \(4x(12 - 2x) = -8x^2 + 48x\). Since the box has maximum volume when \( x = 2 \), the area of the base is \(4(2)^2 - 48(2) + 144 = 64\) square inches and the total area of all four walls is \(-8(2)^2 + 48(2) = 64\) square inches. So, both are 64 square inches.

(b) From Exercise 23, the area of the base is \((3 - 2x)(8 - 2x) = 4x^2 - 22x + 24\) and the total area of all four walls is \(2x(3 - 2x) + 2x(8 - 2x) = -8x^2 + 22x\). Since the box has maximum volume when \( x = \frac{2}{3} \), the area of the base is \(4 \left( \frac{2}{3} \right)^2 - 22 \left( \frac{2}{3} \right) + 24 = \frac{100}{9}\) square feet and the total area of all four walls is \(-8 \left( \frac{2}{3} \right)^2 + 22 \left( \frac{2}{3} \right) = \frac{100}{9}\) square feet. So, both are \(\frac{100}{9}\) square feet.

(c) Based on the results from parts (a) and (b), it appears that the area of the base and the total area of the walls for the box with maximum volume are equal. (This conjecture is true.)

25. Let \( x = \) the width of printed material and \( y = \) the length of printed material.

Then, the area of the printed material is

\[
xy = 36,
\]

so

\[
y = \frac{36}{x}.
\]

Also, \( x + 2 = \) the width of a page and \( y + 3 = \) the length of a page.

The area of a page is

\[
A = (x + 2)(y + 3) = xy + 2y + 3x + 6
\]

\[
= 36 + 2 \left( \frac{36}{x} \right) + 3x + 6
\]

\[
= 42 + \frac{72}{x} + 3x.
\]

\[
A' = -\frac{72}{x^2} + 3 = 0
\]

\[
x = \sqrt{\frac{24}{3}} = 2\sqrt{6}
\]

(We discard \( x = -2\sqrt{6} \) once we must have \( x > 0 \).)

\[
A'' = \frac{216}{x^3} > 0 \text{ when } x = 2\sqrt{6}, \text{ which implies that } A \text{ is minimized when } x = 2\sqrt{6}.
\]

\[
y = \frac{36}{x} = \frac{36}{2\sqrt{6}} = \frac{18}{\sqrt{6}} = \frac{18\sqrt{6}}{6} = 3\sqrt{6}
\]

The width of a page is

\[
x + 2 = 2\sqrt{6} + 2 \approx 6.9 \text{ in.}
\]

The length of a page is

\[
y + 3 = 3\sqrt{6} + 3 \approx 10.3 \text{ in.}
\]

26. Distance on shore: \( 9 - x \) miles

Cost on shore: $400 per mile

Distance underwater: \( \sqrt{x^2 + 36} \)

Cost underwater: $500 per mile

Find the distance from \( A \), that is, \( (9 - x) \), to minimize cost, \( C(x) \).

\[
C(x) = (9 - x)(400) + (\sqrt{x^2 + 36})(500)
\]

\[
= 3600 - 400x + 500(x^2 + 36)^{1/2}
\]

\[
C'(x) = -400 + 500 \left( \frac{1}{2} \right) (x^2 + 36)^{-1/2} (2x)
\]

\[
= -400 + \frac{500x}{\sqrt{x^2 + 36}}
\]
If $C'(x) = 0$, 
\[
\frac{500x}{\sqrt{x^2 + 36}} = 400
\]
\[
\frac{5x}{4} = \sqrt{x^2 + 36}
\]
\[
\frac{25}{16}x^2 = x^2 + 36
\]
\[
\frac{9}{16}x^2 = 36
\]
\[
x^2 = \frac{36 \cdot 16}{9}
\]
\[
x = \frac{6 \cdot 4}{3} = 8.
\]
(Discard the negative solution.)
Then the distance should be 
\[
9 - x = 9 - 8 = 1 \text{ mile from point } A.
\]

27. Distance on shore: $7 - x$ miles
Cost on shore: $\$400$ per mile
Distance underwater: $\sqrt{x^2 + 36}$
Cost underwater: $\$500$ per mile
Find the distance from $A$, that is, $7 - x$, to minimize cost, $C(x)$.
\[
C(x) = (7 - x)(400) + (\sqrt{x^2 + 36})(500)
\]
\[
= 2800 - 400x + 500(x^2 + 36)^{1/2}
\]
\[
C'(x) = -400 + 500 \left( \frac{1}{2} \right) (x^2 + 36)^{-1/2}(2x)
\]
\[
= -400 + \frac{500x}{\sqrt{x^2 + 36}}
\]
If $C'(x) = 0$, 
\[
\frac{500x}{\sqrt{x^2 + 36}} = 400
\]
\[
\frac{5x}{4} = \sqrt{x^2 + 36}
\]
\[
\frac{25}{16}x^2 = x^2 + 36
\]
\[
\frac{9}{16}x^2 = 36
\]
\[
x^2 = \frac{36 \cdot 16}{9}
\]
\[
x = \frac{6 \cdot 4}{3} = 8.
\]
(Discard the negative solution.)
x = 8 is impossible since Point $A$ is only 7 miles from point $C$.

Check the endpoints.
\[
\begin{array}{c|c}
\hline
x & C(x) \\
\hline
0 & 5800 \\
7 & 4610 \\
\hline
\end{array}
\]
The cost is minimized when $x = 7$.
$7 - x = 7 - 7 = 0$, so the company should angle the cable at Point $A$.

28. Let $x =$ the number of additional tables.
Then $160 - 0.50x =$ the cost per table and $250 + x =$ the number of tables ordered.
\[
R = (160 - 0.50x)(250 + x)
\]
\[
= 40,000 + 35x - 0.50x^2
\]
\[
R' = 35 - x = 0
\]
\[
x = 35
\]
$R'' = -1 < 0$, so when $250 + 35 = 285$ tables are ordered, revenue is maximum.
Thus, the maximum revenue is 
\[
R(35) = 40,000 + 35(35) - 0.50(35)^2
\]
\[
= 40,612.5
\]
The maximum revenue is $\$40,612.50$.
Minimum revenue is found by letting $R = 0$.
\[
(160 - 0.50x)(250 + x) = 0
\]
\[
160 - 0.50x = 0 \quad \text{or} \quad 250 + x = 0
\]
\[
x = 320 \quad \text{or} \quad x = -250
\]
\[
(\text{impossible})
\]
So when $250 + 320 = 570$ tables are ordered, revenue is 0, that is, each table is free.
I would fire the assistant.

29. From Example 4, we know that the surface area of the can is given by 
\[
S = 2\pi r^2 + \frac{2000}{r}
\]
Aluminum costs $3\$/cm$^2$, so the cost of the aluminum to make the can is 
\[
0.03 \left( 2\pi r^2 + \frac{2000}{r} \right) = 0.06\pi r^2 + \frac{60}{r}
\]
The perimeter (or circumference) of the circular top is $2\pi r$. Since there is a $2\$/cm charge to seal the top and bottom, the sealing cost is 
\[
0.02(2)(2\pi r) = 0.08\pi r.
\]
Thus, the total cost is given by the function

\[ C(r) = 0.06\pi r^2 + \frac{60}{r} + 0.08\pi r \]

\[ = 0.06\pi r^2 + 60r^{-1} + 0.08\pi r. \]

Then

\[ C'(r) = 0.12\pi r - 60r^{-2} + 0.08\pi \]

\[ = 0.12\pi r - \frac{60}{r^2} + 0.08\pi. \]

Graph

\[ y = 0.12\pi x - \frac{60}{x^2} + 0.08\pi \]

on a graphing calculator. Since \( r \) must be positive in this application, our window should not include negative values of \( x \). A suitable choice for the viewing window is \([0, 10]\) by \([-10, 10]\). From the graph, we find that \( C'(x) = 0 \) when \( x \approx 5.206 \).

Thus, the cost is minimized when the radius is about 5.206 cm.

We can find the corresponding height by using the equation

\[ h = \frac{1000}{\pi r^2} \]

from Example 4.

If \( r = 5.206 \),

\[ h = \frac{1000}{\pi (5.206)^2} \approx 11.75. \]

To minimize cost, the can should have radius 5.206 cm and height 11.75 cm.

**30.** In Exercise 29, we found that the cost of the aluminum to make the can is

\[ 0.03 \left(2\pi r^2 + \frac{2000}{r}\right) = 0.06\pi r^2 + \frac{60}{r}. \]

The cost for the vertical seam is 0.01 \( h \). From Example 4, we see that \( h \) and \( r \) are related by the equation

\[ h = \frac{1000}{\pi r^2}, \]

so the sealing cost is

\[ 0.01h = 0.01 \left(\frac{1000}{\pi r^2}\right) \]

\[ = \frac{1000}{\pi r^2}. \]

Thus, the total cost is given by the function

\[ C(r) = 0.06\pi r^2 + \frac{60}{r} + \frac{10}{\pi r^2} \]

or \( 0.06\pi r^2 + 60r^{-1} + \frac{10}{\pi r^2}. \)

Then

\[ C'(r) = 0.12\pi r - 60r^{-2} - \frac{20}{\pi r^{-3}} \]

or \( 0.12\pi r - \frac{60}{r^2} - \frac{20}{\pi r^3}. \)

Graph

\[ y = 0.12\pi x - \frac{60}{x^2} - \frac{20}{\pi x^3} \]

on a graphing calculator. Since \( r \) must be positive, our window should not include negative values of \( x \). A suitable choice for the viewing window is \([0, 10]\) by \([-10, 10]\). From the graph, we find that \( C'(x) = 0 \) when \( x \approx 5.454 \).

Thus, the cost is minimized when the radius is about 5.454 cm.

We can find the corresponding height by using the equation

\[ h = \frac{1000}{\pi r^2} \]

from Example 4.

If \( r = 5.454 \),

\[ h = \frac{1000}{\pi (5.454)^2} \approx 10.70. \]

To minimize cost, the can should have radius 5.454 cm and height 10.70 cm.

**31.** In Exercises 29 and 30, we found that the cost of the aluminum to make the can is \( 0.06\pi r^2 + \frac{60}{r} \), the cost to seal the top and bottom is \( 0.08\pi r \), and the cost to seal the vertical seam is \( \frac{10}{\pi r^2} \).

Thus, the total cost is now given by the function

\[ C(r) = 0.06\pi r^2 + \frac{60}{r} + 0.08\pi r + \frac{10}{\pi r^2} \]

or \( 0.06\pi r^2 + 60r^{-1} + 0.08\pi r + \frac{10}{\pi r^2}. \)

Then

\[ C'(r) = 0.12\pi r - 60r^{-2} + 0.08\pi - \frac{20}{\pi r^{-3}} \]

or \( 0.12\pi r - \frac{60}{r^2} + 0.08\pi - \frac{20}{\pi r^3}. \)

Graph

\[ y = 0.12\pi x - \frac{60}{x^2} + 0.08\pi - \frac{20}{\pi x^3} \]
on a graphing calculator. A suitable choice for the viewing window is \([0, 10] \times [-10, 10]\). From the graph, we find that \(C'(x) = 0\) when \(x \approx 5.242\).

Thus, the cost is minimized when the radius is about 5.242 cm.

To find the corresponding height, use the equation

\[
h = \frac{1000}{\pi r^2}
\]

from Example 4.

If \(r = 5.242\),

\[
h = \frac{1000}{\pi (5.242)^2} \approx 11.58.
\]

To minimize cost, the can should have radius 5.242 cm and height 11.58 cm.

32. \(p(t) = \frac{20t^3 - t^4}{1000}, \ [0, 20]\)

(a) \(p'(t) = \frac{3}{50}t^2 - \frac{1}{250}t^3\)

\[
= \frac{1}{50}t^2 \left[ 3 - \frac{1}{5}t \right]
\]

Critical numbers:

\[
\frac{1}{50}t^2 = 0 \quad \text{or} \quad 3 - \frac{1}{5}t = 0
\]

\[
t = 0 \quad \text{or} \quad t = 15
\]

The number of people infected reaches a maximum in 15 days.

(b) \(P(15) = 16.875\%\)

33. \(N(t) = 20 \left( \frac{t}{12} - \ln \left( \frac{t}{12} \right) \right) + 30; \ 1 \leq t \leq 15\)

\[\]

\[
N'(t) = 20 \left[ \frac{1}{12} - \frac{1}{t} \right]
\]

\[
= 20 \left( \frac{1}{12} - \frac{1}{t} \right)
\]

\[
= \frac{20(t - 12)}{12t}
\]

\(N'(t) = 0\) when

\[
t - 12 = 0
\]

\[
t = 12.
\]

\(N''(t)\) does not exist at \(t = 0\), but 0 is not in the domain of \(N\).

Thus, 12 is the only critical number.

To find the absolute extrema on \([1, 15]\), evaluate \(N\) at the critical number and at the endpoints.

<table>
<thead>
<tr>
<th>(t)</th>
<th>(N(t))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>81.365</td>
</tr>
<tr>
<td>12</td>
<td>50</td>
</tr>
<tr>
<td>15</td>
<td>50.537</td>
</tr>
</tbody>
</table>

Use this table to answer the questions in (a)-(d).

(a) The number of bacteria will be a minimum at \(t = 12\), which represents 12 days.

(b) The minimum number of bacteria is given by \(N(12) = 50\), which represents 50 bacteria per ml.

(c) The number of bacteria will be a maximum at \(t = 1\), which represents 1 day.

(d) The maximum number of bacteria is given by \(N(1) = 81.365\), which represents 81.365 bacteria per ml.

34. (a) \(p(t) = 10e^{-t/8}, \ [0, 40]\)

\[p'(t) = 10e^{-t/8} \left( -\frac{1}{8} \right) + e^{-t/8}(10)\]

\[
= 10e^{-t/8} \left( -\frac{t}{8} + 1 \right)
\]

Critical numbers:

\(p'(t) = 0\) when

\[
-\frac{t}{8} + 1 = 0
\]

\[
t = 8.
\]

<table>
<thead>
<tr>
<th>(t)</th>
<th>(p(t))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>29.43</td>
</tr>
<tr>
<td>40</td>
<td>2.6952</td>
</tr>
</tbody>
</table>

The percent of the population infected reaches a maximum in 8 days.

(b) \(P(8) = 29.43\%\)
35. \( H(S) = f(S) - S \)
\[
\begin{align*}
    f(S) &= 125e^{0.25} \\
    H(S) &= 125e^{0.25} - S \\
    H'(S) &= 3S^{-0.75} - 1 \\
    H'(0) &= 0 \\
    3S^{-0.75} - 1 &= 0 \\
    S^{-0.75} &= \frac{1}{3} \\
    S^{0.75} &= 3 \\
    S^{3/4} &= 3 \\
    S &= 3^{4/3} \\
    S &= 4.327.
\end{align*}
\]

The maximum sustainable harvest is 12.86 thousand.

36. \( H(S) = f(S) - S \)
\[
\begin{align*}
    f(S) &= \frac{25S}{S + 2} \\
    H(S) &= \frac{(S + 2)(25) - 25S}{(S + 2)^2} - 1 \\
    &= \frac{25S + 50 - 25S - (S + 2)^2}{(S + 2)^2} \\
    &= \frac{50 - (S^2 + 4S + 4)}{(S + 2)^2} \\
    &= \frac{-S^2 - 4S + 46}{(S + 2)^2} \\
    H'(S) &= 0 \\
    S^2 + 4S - 46 &= 0 \\
    S &= \frac{-4 \pm \sqrt{16 + 184}}{2} \\
    &= 5.071.
\end{align*}
\]

(Discard the negative solution.)

The number of creatures needed to sustain the population is \( S_0 = 5.071 \) thousand.

\[
H''(S) = \frac{(S+2)^2(-2S-4)-(S^2-4S+46)(2S+4)}{(S+2)^4} < 0,
\]

so \( H \) is a maximum at \( S_0 = 5.071 \).

\[
H(S_0) = \frac{25(5.071)}{7.071} - 5.071 \\
\approx 12.86
\]

The maximum sustainable harvest is 12.86 thousand.

37. (a) \( H(S) = f(S) - S \)
\[
\begin{align*}
    f(S) &= Se^{r(1-S/P)} \\
    H'(S) &= Se^{r(1-S/P)} \left( -\frac{r}{P} \right) + e^{r(1-S/P)} - 1 \\
    &= Se^{r(1-S/P)} \left( -\frac{r}{P} \right) + e^{r(1-S/P)} = 1 \\
    f'(S) &= 1
\end{align*}
\]

(b) \( f(S) = Se^{r(1-S/P)} \)
\[
\begin{align*}
    f'(S) &= \left( -\frac{r}{P} \right) Se^{r(1-S/P)} + e^{r(1-S/P)} \\
    f'(S_0) &= 1 \\
    e^{r(1-S_0/P)} \left[ -\frac{rS_0}{P} + 1 \right] &= 1 \\
    e^{r(1-S_0/P)} &= \frac{1}{-\frac{rS_0}{P} + 1}
\end{align*}
\]

Using \( H(S) \) from part (a), we get

\[
\begin{align*}
    H(S_0) &= S_0 e^{r(1-S_0/P)} - S_0 \\
    &= S_0 (e^{r(1-S_0/P)} - 1) \\
    &= S_0 \left( \frac{1}{1 - \frac{rS_0}{P}} - 1 \right).
\end{align*}
\]

38. \( r = 0.1, \ P = 100 \)
\[
\begin{align*}
    f(S) &= Se^{r(1-S/P)} \\
    f'(S) &= -\frac{1}{100} Se^{0.1(1-S/100)} + e^{0.1(1-S/100)} \\
    f'(S_0) &= -0.001Se^{0.1(1-S_0/100)} + e^{0.1(1-S_0/100)} \\
    Graph \hspace{1cm} Y_1 &= -0.001xe^{0.1(1-x/100)} + e^{0.1(1-x/100)}
\end{align*}
\]
and
\[ Y_2 = 1 \]
on the same screen. A suitable choice for the viewing window is \([0, 60]\) by \([0.5, 1.5]\) with \(X_{\text{scl}} = 10\) and \(Y_{\text{scl}} = 0.5\). By zooming or using the “intersect” option, we find the graphs intersect when \(x \approx 49.37\).
The maximum sustainable harvest is 49.37.

39. \(r = 0.4, P = 500\)
\[ f(S) = S e^{r(1-S/P)} \]
\[ f’(S) = -\frac{0.4}{500} S e^{0.4(1-S/500)} + e^{0.4(1-S/500)} \]
\[ f’(S_0) = -0.0008 S_0 e^{0.4(1-S_0/500)} + e^{0.4(1-S_0/500)} \]
Graph
\[ Y_1 = -0.0008 S_0 e^{0.4(1-x/500)} + e^{0.4(1-x/500)} \]
and
\[ Y_2 = 1 \]
on the same screen. A suitable choice for the viewing window is \([0, 300]\) by \([0.5, 1.5]\) with \(X_{\text{scl}} = 50\), \(Y_{\text{scl}} = 0.5\). By zooming or using the “intersect” option, we find that the graphs intersect when \(x \approx 237.10\).
The maximum sustainable harvest is 237.10.

40. Let \(x = \text{distance from } P \text{ to } A\).

Energy used over land: 1 unit per mile
Energy used over water: \(\frac{4}{3}\) units per mile
Distance over land: \((2-x)\) mi
Distance over water: \(\sqrt{1+x^2}\) mi
Find the location of \(P\) to minimize energy used.
\[ E(x) = 1(2-x) + \frac{4}{3} \sqrt{1+x^2}, \text{ where } 0 \leq x \leq 2. \]
\[ E’(x) = -1 + \frac{4}{3} \left( \frac{1}{2} \right) (1+x^2)^{-1/2}(2x) \]

If \(E’(x) = 0\),
\[ \frac{4}{3} x(1+x^2)^{-1/2} = 1 \]
\[ \frac{4x}{3(1+x^2)^{1/2}} = 1 \]
\[ \frac{4x}{3(1+x^2)^{1/2}} = (1+x^2)^{1/2} \]
\[ \frac{4x}{3} = 1 + x^2 \]
\[ x^2 = \frac{9}{7} \]
\[ x = \frac{3}{\sqrt{7}} \]
\[ P \text{ is } \frac{3\sqrt{7}}{7} \text{ mi from } A. \]

41. Let \(x = \text{distance from } P \text{ to } A\).

Energy used over land: 1 unit per mile
Energy used over water: \(\frac{10}{9}\) units per mile
Distance over land: \((2-x)\) mi
Distance over water: \(\sqrt{1+x^2}\) mi
Find the location of \(P\) to minimize energy used.
\[ E(x) = 1(2-x) + \frac{10}{9} \sqrt{1+x^2}, \text{ where } 0 \leq x \leq 2. \]
\[ E’(x) = -1 + \frac{10}{9} \left( \frac{1}{2} \right) (1+x^2)^{-1/2}(2x) \]
If \(E’(x) = 0\),
\[ \frac{10}{9} x(1+x^2)^{-1/2} = 1 \]
\[ \frac{10x}{9(1+x^2)^{1/2}} = 1 \]
\[
\frac{10}{9} x = (1 + x^2)^{1/2} \\
\frac{100}{81} x^2 = 1 + x^2 \\
\frac{19}{81} x^2 = 1 \\
x^2 = \frac{81}{19} \\
x = \frac{9}{\sqrt{19}} \\
= \frac{9\sqrt{19}}{19} \\
\approx 2.06.
\]

This value cannot give the absolute maximum since the total distance from \(A\) to \(L\) is just 2 miles. Test the endpoints of the domain.

\[\begin{array}{c|c}
x & E(x) \\
\hline
0 & 3.1416 \approx 3.1111 \\
2 & 2.4845
\end{array}\]

Point \(P\) must be at Point \(L\).

**42. (a)** \(f(S) = aS e^{-bS}\) \(f(S) = Se^{(1-S/P)} \equiv Se^{rS/P} \equiv Se^{-rS/P} \equiv e^{r}Se^{-(r/P)S}\)

Comparing the two terms, replace \(a\) with \(e^r\) and \(b\) with \(r/P\).

**b)** Shepherd:

\[
\begin{align*}
f(S) &= \frac{aS}{1 + (S/b)^c} \\
f'(S) &= \frac{\left[1 + (S/b)^c\right](a) - aS \left[c(S/b)^{c-1}(1/b)\right]}{\left[1 + (S/b)^{2c}\right]^2} \\
&= \frac{a + a(S/b)^c - ac(S/b)(S/b)^{c-1}}{\left[1 + (S/b)^{2c}\right]^2} \\
&= \frac{a + a(S/b)^c - ac(S/b)^c}{\left[1 + (S/b)^{2c}\right]^2} \\
&= \frac{a[1 + (1 - c)(S/b)^c]}{\left[1 + (S/b)^{2c}\right]^2}
\end{align*}
\]

Ricker:

\[
\begin{align*}
f(S) &= aSe^{-bS} \\
f'(S) &= ae^{-bS} + aSe^{-bS}(-b) \\
&= ae^{-bS}(1 - bS)
\end{align*}
\]

Beverton-Holt:

\[
\begin{align*}
f(S) &= \frac{aS}{1 + (S/b)} \\
f'(S) &= \frac{\left[1 + (S/b)\right](a) - aS(1/b)}{\left[1 + (S/b)^2\right]^2} \\
&= \frac{a + a(S/b) - a(S/b)}{\left[1 + (S/b)^2\right]^2} \\
&= \frac{a}{\left[1 + (S/b)^2\right]^2}
\end{align*}
\]

(c) Shepherd:

\[
f'(0) = \frac{a[1 + (1 - c)(0/b)^c]}{\left[1 + (0/b)^c\right]^2} = a
\]

Ricker:

\[
f'(0) = ae^{-b(0)}[1 - b(0)] = a
\]

Beverton-Holt:

\[
f'(0) = \frac{a}{\left[1 + (0/b)^c\right]^2} = a
\]

The constant \(a\) represents the slope of the graph of \(f(S)\) at \(S = 0\).

(d) First find the critical numbers by solving \(f'(S) = 0\).

Shepherd:

\[
f'(S) = 0 \\
a[1 + (1 - c)(S/b)^c] = 0 \\
(1 - c)(S/b)^c = -1 \\
(c - 1)(S/b)^c = 1
\]

Substitute \(b = 248.72\) and \(c = 3.24\) and solve for \(S\).

\[
(3.24 - 1)(S/248.72)^{3.24} = 1 \\
\left(\frac{S}{248.72}\right)^{3.24} = \frac{1}{2.24} \\
\frac{S}{248.72} = \left(\frac{1}{2.24}\right)^{1/3.24} \\
S = 248.72 \left(\frac{1}{2.24}\right)^{1/3.24} \\
S \approx 193.914
\]
Using the Shepherd model, next year’s population is maximized when this year’s population is about 194,000 tons. This can be verified by examining the graph of \( f(S) \).

**(e)** First find the critical numbers by solving \( f'(S) = 0 \).

Ricker:

\[
\begin{align*}
f'(S) &= 0 \\
ae^{-bS}(1-bS) &= 0 \\
1 - bS &= 0 \\
bS &= 1 \\
S &= \frac{1}{b}
\end{align*}
\]

Substitute \( b = 0.0039 \) and solve for \( S \):

\[
S = \frac{1}{0.0039} \\
S \approx 256.410
\]

Using the Ricker model, next year’s population is maximized when this year’s population is about 256,000 tons. This can be verified by examining the graph of \( f(S) \).

43. (a) Solve the given equation for effective power for \( T \), time.

\[
\frac{kE}{T} = aSv^3 + I \\
\frac{kE}{aSv^3 + I} = T
\]

Since distance is velocity, \( v \), times time, \( T \), we have

\[
D(v) = v \frac{kE}{aSv^3 + I} = \frac{kEv}{aSv^3 + I}.
\]

(b) \( D'(v) = \frac{(aSv^3 + I)kE - kEv(3aSv^2)}{(aSv^3 + I)^2} \)

\[
= \frac{kE(aSv^3 + I - 3aSv^3)}{(aSv^3 + I)^2} \\
= \frac{kE(I - 2aSv^3)}{(aSv^3 + I)^2}
\]

Find the critical numbers by solving \( D'(v) = 0 \) for \( v \).

\[
\begin{align*}
I - 2aSv^3 &= 0 \\
2aSv^3 &= I \\
v^3 &= \frac{I}{2aS} \\
v &= \left(\frac{I}{2aS}\right)^{1/3}
\end{align*}
\]

44. Let \( x = \) width. Then \( x = \) height

\[
\text{and } 108 - 4x = \text{length}.
\]

(since length plus girth = 108)

\[
\begin{align*}
V(x) &= l \cdot w \cdot h \\
&= (108 - 4x)x \cdot x \\
&= 108x^2 - 4x^3 \\
V'(x) &= 216x - 12x^2
\end{align*}
\]

Set \( V'(x) = 0 \), and solve for \( x \).

\[
216x - 12x^2 = 0 \\
12x(18 - x) = 0 \\
x = 0 \quad \text{or} \quad x = 18
\]

0 is not in the domain, so the only critical number is 18.

Width = 18
Height = 18
Length = 108 - 4(18) = 36

The dimensions of the box with maximum volume are 36 inches by 18 inches by 18 inches.

45. Let \( 8 - x = \) the distance the hunter will travel on the river.

Then \( \sqrt{9 + x^2} = \) the distance he will travel on land.
The rate on the river is 5 mph, the rate on land is 2 mph. Using $t = \frac{d}{r}$,

\[
\frac{8-x}{5} = \text{the time on the river},
\]

\[
\frac{\sqrt{9+x^2}}{2} = \text{the time on land}.
\]

The total time is

\[
T(x) = \frac{8-x}{5} + \frac{\sqrt{9+x^2}}{2} = \frac{8}{5} - \frac{x}{5} + \frac{1}{2}(9+x^2)^{1/2}.
\]

\[
T'(x) = -\frac{1}{5} + \frac{1}{4} \cdot 2x(9+x^2)^{-1/2}
\]

\[
-\frac{1}{5} + \frac{x}{2(9+x^2)^{1/2}} = 0
\]

\[
\frac{1}{5} = \frac{x}{2(9+x^2)^{1/2}}
\]

\[
2(9+x^2)^{1/2} = 5x
\]

\[
4(9+x^2) = 25x^2
\]

\[
36 + 4x^2 = 25x^2
\]

\[
36 = 21x^2
\]

\[
\frac{6}{2\sqrt{21}} = x
\]

\[
\frac{6\sqrt{21}}{21} = \frac{2\sqrt{21}}{7} = x
\]

\[
\begin{array}{c|c}
 x & T(x) \\
 \hline
 0 & 3.1 \\
 2\sqrt{21} & 2.98 \\
 8 & 4.27 \\
\end{array}
\]

Since the minimum time is 2.98 hr, the hunter should travel $8 - \frac{2\sqrt{21}}{7}$ or about 6.7 miles along the river.

46. Let $8-x$ = the distance the hunter will travel on the river.

Then $\sqrt{19^2 + x^2}$ = the distance he will travel on land.

Since the rate on the river is 5 mph, the rate on land is 2 mph, and $t = \frac{d}{r}$,

\[
\frac{8-x}{5} = \text{the time on the river}
\]

\[
\frac{\sqrt{361 + x^2}}{2} = \text{the time on the land}.
\]

The total time is

\[
T(x) = \frac{8-x}{5} + \frac{\sqrt{361 + x^2}}{2} = \frac{8}{5} - \frac{x}{5} + \frac{1}{2}(361+x^2)^{1/2}.
\]

\[
T'(x) = -\frac{1}{5} + \frac{1}{4} \cdot 2(361+x^2)^{-1/2}
\]

\[
-\frac{1}{5} + \frac{x}{2(361+x^2)^{1/2}} = 0
\]

\[
\frac{1}{5} = \frac{x}{2(361+x^2)^{1/2}}
\]

\[
2(361+x^2)^{1/2} = 5x
\]

\[
4(361+x^2) = 25x^2
\]

\[
1444 + 4x^2 = 25x^2
\]

\[
1444 = 21x^2
\]

\[
\frac{38}{\sqrt{21}} = x
\]

\[
\frac{8.29}{x} = 8 \text{ miles}
\]

8.29 is not possible, since the cabin is only 8 miles west. Check the endpoints.

\[
\begin{array}{c|c}
 x & T(x) \\
 \hline
 0 & 11.1 \\
 8 & 10.3 \\
\end{array}
\]

$T(x)$ is minimized when $x = 8$.

The distance along the river is given by $8-x$, so the hunter should travel $8 - 8 = 0$ miles along the river. He should complete the entire trip on land.
6.3 Further Business Applications: Economic Lot Size; Economic Order Quantity; Elasticity of Demand

1. When \( q < \sqrt{\frac{2fM}{k}} \), \( T'(q) < -\frac{k}{2} + \frac{k}{q} = 0 \); and when \( q > \sqrt{\frac{2fM}{k}} \), \( T'(q) > -\frac{k}{2} + \frac{k}{q} = 0 \). Since the function \( T(q) \) is decreasing before \( q = \sqrt{\frac{2fM}{k}} \) and increasing after \( q = \sqrt{\frac{2fM}{k}} \), there must be a relative minimum at \( q = \sqrt{\frac{2fM}{k}} \). By the critical point theorem, there is an absolute minimum there.

3. The economic order quantity formula assumes that \( M \), the total units needed per year, is known. Thus, \( c \) is the correct answer.

4. Use equation (3) with \( k = 1 \), \( M = 100,000 \), and \( f = 500 \).
\[
q = \sqrt{\frac{2fM}{k}} = \sqrt{\frac{2(500)(100,000)}{1}} = \sqrt{100,000,000} = 10,000
\]
10,000 lamps should be made in each batch to minimize production costs.

5. Use equation (3) with \( k = 9 \), \( M = 13,950 \), and \( f = 31 \).
\[
q = \sqrt{\frac{2fM}{k}} = \sqrt{\frac{2(31)(13,950)}{9}} = \sqrt{96,100} = 310
\]
310 cases should be made in each batch to minimize production costs.

6. From Exercise 4, \( M = 100,000 \), and \( q = 10,000 \). The number of batches per year is \( \frac{M}{q} = \frac{100,000}{10,000} = 10 \).

7. From Exercise 5, \( M = 13,950 \) and \( q = 310 \). The number of batches per year is
\[
\frac{M}{q} = \frac{13,950}{310} = 45
\]
45 cases should be made in each batch to minimize production costs.

8. Here \( k = 0.50 \), \( M = 100,000 \), and \( f = 60 \). We have
\[
q = \sqrt{\frac{2fM}{k}} = \sqrt{\frac{2(60)(100,000)}{0.50}} = \sqrt{24,000,000} \approx 4898.98
\]
\( T(4898) = 2449.489792 \) and \( T(4899) = 2449.489743 \), so ordering 4899 copies per order minimizes the annual costs.

9. Here \( k = 1 \), \( M = 900 \), and \( f = 5 \). We have
\[
q = \sqrt{\frac{2fM}{k}} = \sqrt{\frac{2(5)(900)}{1}} = \sqrt{9000} \approx 94.9
\]
\( T(94) \approx 94.872 \) and \( T(95) \approx 94.868 \), so ordering 95 bottles per order minimizes the annual costs.

10. Using maximum inventory size,
\[
T(q) = \frac{fM}{q} + gM + kq; \ (0, \infty)
\]
\[
T'(q) = -\frac{fM}{q^2} + k
\]
Set the derivative equal to 0.
\[
-\frac{fM}{q^2} + k = 0
\]
\[
k = \frac{fM}{q^2}
\]
\[
q^2k = fM
\]
\[
q^2 = \frac{fM}{k}
\]
\[
q = \sqrt{\frac{fM}{k}}
\]
Since \( \lim_{q \to 0} T(q) = \infty \), \( \lim_{q \to \infty} T(q) = \infty \), and \( q = \sqrt{\frac{fM}{k}} \) is the only critical value in \( (0, \infty) \), \( q = \sqrt{\frac{fM}{k}} \) is the number of units that should be ordered or manufactured to minimize total costs.
11. Use \( q = \sqrt{\frac{fM}{k}} \) from Exercise 10 with \( k = 6, M = 5000, \) and \( f = 1000. \)

\[
q = \sqrt{\frac{fM}{k}} = \sqrt{\frac{(1000)(5000)}{6}} \approx 912.9
\]

with \( T(q) = \frac{fM}{q} + kq \) (assume \( g = 0 \) since the subsequent cost per book is so low that it can be ignored), \( T(912) \approx 10,954.456 \) and \( T(913) \approx 10,954.451. \) So, 913 books should be printed in each print run.

12. Assuming an annual cost, \( k_1, \) for storing a single unit, plus an annual cost per unit, \( k_2, \) that must be paid for each unit up to the maximum number of units stored, we have

\[
T(q) = \frac{fM}{q} + gM + \frac{k_1 q}{2} + k_2q; \quad (0, \infty)
\]

\[
T'(q) = -\frac{fM}{q^2} + \frac{k_1}{2} + k_2
\]

Set this derivative equal to 0.

\[
-\frac{fM}{q^2} + \frac{k_1}{2} + k_2 = 0
\]

\[
\frac{k_1}{2} + k_2 = \frac{fM}{q^2}
\]

\[
\frac{k_1 + 2k_2}{2} = \frac{fM}{q^2}
\]

\[
\frac{q^2(k_1 + 2k_2)}{2} = fM
\]

\[
q^2 = \frac{2fM}{k_1 + 2k_2}
\]

\[
q = \sqrt{\frac{2fM}{k_1 + 2k_2}}
\]

Since \( \lim_{q \to 0} T(q) = \infty, \lim_{q \to \infty} T(q) = \infty, \) and

\[
q = \sqrt{\frac{2fM}{k_1 + 2k_2}}
\]

is the only critical value in \((0, \infty),\)

\[
q = \sqrt{\frac{2fM}{k_1 + 2k_2}}
\]

is the number of units that should be ordered or manufactured to minimize the total cost in this case.

13. Use \( q = \sqrt{\frac{2fM}{k_1 + 2k_2}} \) from Exercise 12 with \( k_1 = 1, \)

\( k_2 = 2, M = 30,000, \) and \( f = 750. \) Also, note that \( g = 8. \)

\[
q = \sqrt{\frac{2fM}{k_1 + 2k_2}}
\]

\[
= \sqrt{\frac{2(750)(30,000)}{1 + 2(2)}}
\]

\[
= \sqrt{9,000,000} = 3000
\]

The number of production runs each year to minimize her total costs is

\[
\frac{M}{q} = \frac{30,000}{3000} = 10.
\]

14. \( q = 25,000 - 50p \)

(a) \( \frac{dq}{dp} = -50 \)

\[
E = -p \cdot \frac{dq}{dp}
\]

\[
= -p \cdot \frac{-50}{500 - p}
\]

\[
= \frac{p}{500 - p}
\]

(b) \( R = pq \)

\[
\frac{dR}{dp} = q(1 - E)
\]

When \( R \) is maximum, \( q(1 - E) = 0. \)

Since \( q = 0 \) means no revenue, set \( 1 - E = 0. \)

\[
E = 1
\]

From part (a),

\[
\frac{p}{500 - p} = 1
\]

\[
p = 500 - p
\]

\[
p = 250.
\]

\[
q = 25,000 - 50p
\]

\[
= 25,000 - 50(250)
\]

\[
= 12,500
\]

Total revenue is maximized if \( q = 12,500. \)
15. \( q = 50 - \frac{p}{4} \)

(a) \( \frac{dq}{dp} = -\frac{1}{4} \)

\[
E = \frac{p}{q} \cdot \frac{dq}{dp}
\]
\[
= \frac{p}{50 - \frac{p}{4}} \left( -\frac{1}{4} \right)
\]
\[
= \frac{p}{200 - p} \left( -\frac{1}{4} \right)
\]

(b) \( R = pq \)

\[ \frac{dR}{dp} = q(1 - E) \]

When \( R \) is maximum, \( q(1 - E) = 0 \). Since \( q = 0 \) means no revenue, set \( 1 - E = 0 \).

\[ E = 1 \]

From part (a),

\[
\frac{2p^2}{4800 - p^2} = 1
\]
\[
2p^2 = 4800 - p^2
\]
\[
3p^2 = 4800
\]
\[
p^2 = 1600
\]
\[
p = \pm 40.
\]

Since \( p \) must be positive, \( p = 40 \).

\[ q = 48,000 - 10p^2 \]
\[ = 48,000 - 10(40^2) \]
\[ = 48,000 - 10(1600) \]
\[ = 48,000 - 16,000 \]
\[ = 32,000 \]

16. \( q = 48,000 - 10p^2 \)

(a) \( \frac{dq}{dp} = -20p \)

\[
E = \frac{-p}{q} \cdot \frac{dq}{dp}
\]
\[
= \frac{-p}{48,000 - 10p^2}(-20p)
\]
\[
= \frac{20p^2}{48,000 - 10p^2}
\]
\[
= \frac{2p^2}{4800 - p^2}
\]

(b) \( R = pq \)

\[ \frac{dR}{dp} = q(1 - E) \]

Total revenue is maximized if \( q = 25 \).

17. (a) \( q = 37,500 - 5p^2 \)

\[ \frac{dq}{dp} = -10p \]

\[
E = \frac{-p}{q} \cdot \frac{dq}{dp}
\]
\[
= \frac{-p}{37,500 - 5p^2}(-10p)
\]
\[
= \frac{10p^2}{37,500 - 5p^2}
\]
\[
= \frac{2p^2}{7500 - p^2}
\]

(b) \( R = pq \)

\[ \frac{dR}{dp} = q(1 - E) \]

When \( R \) is maximum, \( q(1 - E) = 0 \). Since \( q = 0 \) means no revenue, set \( 1 - E = 0 \).

\[ E = 1 \]

From (a),

\[
\frac{2p^2}{7500 - p^2} = 1
\]
\[
2p^2 = 7500 - p^2
\]
\[
3p^2 = 7500
\]
\[
p^2 = 2500
\]
\[
p = \pm 50.
\]
Since \( p \) must be positive, \( p = 50 \).

\[
q = 37,500 - 5p^2 \\
= 37,500 - 5(50)^2 \\
= 37,500 - 5(2500) \\
= 37,500 - 12,500 \\
= 25,000.
\]

18. \( q = 10 - \ln p \)

(a) \( \frac{dq}{dp} = -\frac{1}{p} \)

\[
E = -\frac{p}{q} \cdot \frac{dq}{dp} \\
= -\frac{p}{10 - \ln p} \left( -\frac{1}{p} \right) \\
= \frac{1}{10 - \ln p}
\]

(b) \( R = pq \)

\[
\frac{dR}{dp} = q(1 - E)
\]

When \( R \) is maximum, \( q(1 - E) = 0 \). Since \( q = 0 \) means no revenue, set \( 1 - E = 0 \).

\[
E = 1
\]

From part (a),

\[
\frac{1}{10 - \ln p} = 1 \\
10 - \ln p = 1 \\
\ln p = 9 \\
p = e^9 \\
q = 10 - \ln p \\
= 10 - \ln e^9 \\
= 10 - 9 \\
= 1
\]

Note that \( E = \frac{1}{q} \), thus we would expect \( E \) to be maximum when \( q = 1 \).

19. \( p = 400e^{-0.2q} \)

In order to find the derivative \( \frac{dp}{dp} \), we first need to solve for \( q \) in the equation \( p = 400e^{-0.2q} \).

(a) \( \frac{p}{400} = e^{-0.2q} \)

\[
\ln \left( \frac{p}{400} \right) = \ln (e^{-0.2q}) = -0.2q \\
q = -5 \ln \left( \frac{p}{400} \right)
\]

Now

\[
\frac{dq}{dp} = -5 \frac{1}{\frac{400}{p}} = -5 \frac{p}{400}, \text{ and} \\
E = -\frac{p}{q} \cdot \frac{dq}{dp} = -\frac{p}{q} \cdot -\frac{5}{q} = \frac{5}{q}
\]

(b) \( R = pq \)

\[
\frac{dR}{dp} = q(1 - E)
\]

When \( R \) is maximum, \( q(1 - E) = 0 \). Since \( q = 0 \) means no revenue, set \( 1 - E = 0 \).

\[
E = 1
\]

From part (a),

\[
\frac{5}{q} = 1 \\
5 = q
\]

20. \( q = 300 - 2p \)

\[
\frac{dq}{dp} = -2 \\
E = -\frac{p}{q} \cdot \frac{dq}{dp} \\
E = -\frac{p(-2)}{300 - 2p} \\
= \frac{2p}{300 - 2p}
\]

(a) When \( p = $100 \),

\[
E = \frac{200}{300 - 200} \\
= 2
\]

Since \( E > 1 \), demand is elastic. This indicates that a percentage increase in price will result in a greater percentage decrease in demand.

(b) When \( p = $50 \),

\[
E = \frac{100}{300 - 100} \\
= \frac{1}{2} < 1
\]

Since \( E < 1 \), supply is inelastic. This indicates that a percentage change in price will result in a smaller percentage change in demand.
21. \( q = 400 - 0.2p^2 \)
\[
\frac{dq}{dp} = 0 - 0.4p
\]
\[
E = -\frac{p \cdot \frac{dq}{dp}}{q} = -\frac{p^2}{400 - 0.2p^2} \cdot (-0.4p)
\]
\[
E = \frac{0.4p^2}{400 - 0.2p^2}
\]

(a) If \( p = \$20 \),
\[
E = \frac{(0.4)(20)^2}{400 - 0.2(20)^2} = 0.5
\]
Since \( E < 1 \), demand is inelastic. This indicates that total revenue increases as price increases.

(b) If \( p = \$40 \),
\[
E = \frac{(0.4)(40)^2}{400 - 0.2(40)^2} = 8
\]
Since \( E > 1 \), demand is elastic. This indicates that total revenue decreases as price increases.

22. \( q = 342.5p^{-0.5314} \)
\[
\frac{dq}{dp} = 342.5(-0.5314)p^{-0.5314-1}
\]
\[
= \frac{-182}{p^{1.5314}}
\]
\[
E = -\frac{p \cdot \frac{dq}{dp}}{q} = -\frac{-p}{342.5p^{-0.5314} \cdot p^{1.5314}} \cdot \frac{-182}{p^{1.5314}}
\]
\[
= \frac{182}{342.5} \approx 0.5314
\]
Since \( E < 1 \), the demand is inelastic.

23. (a) \( q = 55.2 - 0.022p \)
\[
\frac{dq}{dp} = -0.022
\]
\[
E = -\frac{p \cdot \frac{dq}{dp}}{q} = -\frac{-p}{55.2 - 0.022p} \cdot (-0.022)
\]
\[
= \frac{0.022p}{55.2 - 0.022p}
\]
When \( p = \$166.10 \),
\[
E = \frac{3.6542}{55.2 - 3.6542} \approx 0.071
\]

(b) Since \( E < 1 \), the demand for airfare is inelastic at this price.

(c) \( R = pq \)
\[
\frac{dR}{dp} = q(1 - E)
\]
When \( R \) is a maximum, \( q(1 - E) = 0 \).
Since \( q = 0 \) means no revenue, set \( 1 - E = 0 \).
\[
E = 1
\]
From (a),
\[
\frac{0.022p}{55.2 - 0.022p} = 1
\]
\[
0.022p = 55.2 - 0.022p
\]
\[
0.044p = 55.2
\]
\[
p \approx 1255
\]
Total revenue is maximized if \( p \approx \$1255 \).

24. \( p = 0.604q^2 - 20.16q + 263.067 \)
\[
\frac{dp}{dq} = 1.208q - 20.16
\]
\[
E = -\frac{p}{q} \cdot \frac{dp}{dq} = -\frac{p}{q} \cdot \frac{1.208q - 20.16}{1.208q - 20.16}
\]
\[
= \frac{0.604q^2 - 20.16q + 263.067}{-q(1.208q - 20.16)}
\]

(a) Since \( q = 11 \),
\[
E = \frac{114.391}{-11(-6.872)} \approx 1.51
\]

(b) Since \( E > 1 \), demand is elastic.

(c) As \( q \) approaches 16.6887, the denominator of \( E \) approaches zero, so \( E \) approaches infinity.
25. \( q = m - np \) for \( 0 \leq p \leq \frac{m}{n} \)

\[
\frac{dq}{dp} = -n
\]

\( E = \frac{-p}{q} \cdot \frac{dq}{dp} \)

\( E = -\frac{p}{m - np}(-n) \)

\( E = \frac{pn}{m - np} = 1 \)

\[
pn = m - np \\
2np = m \\
p = \frac{m}{2n}
\]

Thus, \( E = 1 \) when \( p = \frac{m}{2n} \), or at the midpoint of the demand curve on the interval \( 0 \leq p \leq \frac{m}{n} \).

26. \( E = \frac{-p}{q} \cdot \frac{dq}{dp} \)

Since \( p \neq 0 \), \( E = 0 \) when \( \frac{dq}{dp} = 0 \). The derivative is zero, which implies that the demand function has a horizontal tangent line at the value of \( p \) where \( E = 0 \).

27. (a) \( q = Cp^{-k} \)

\[
\frac{dq}{dp} = -Ckp^{-k-1}
\]

\( E = \frac{-p}{q} \cdot \frac{dq}{dp} \)

\[\begin{align*}
E &= \frac{-p}{Cp^{-k}}(-Ckp^{-k-1}) \\
E &= \frac{k}{p^{-k}} = k
\end{align*}\]

28. \( \frac{dq}{dp} = -\frac{BR}{RP} = \frac{OB - OR}{-RP} = \frac{OB - q_0}{-p_0} \)

or

\[\begin{align*}
-p_0 \frac{dq}{dp} &= OB - q_0 \\
-p_0 \frac{dq}{q_0} &= \frac{OB}{q_0} - 1 \\
&= \frac{OB}{OR} - 1
\end{align*}\]

Because triangles \( AOB \) and \( PRB \) are similar,

\[\begin{align*}
-p_0 \cdot \frac{dq}{q_0} &= \frac{AB}{AP} - 1 \\
&= \frac{PB}{PA}
\end{align*}\]

But \( E = -\frac{p_0}{q_0} \cdot \frac{dq}{dp} \) so the ratio \( \frac{PB}{PA} \) equals the elasticity \( E \).

29. The demand function \( q(p) \) is positive and increasing, so \( \frac{dq}{dp} \) is positive. Since \( p_0 \) and \( q_0 \) are also positive, the elasticity \( E = -\frac{p_0}{q_0} \cdot \frac{dq}{dp} \) is negative.
Chapter 6  APPLICATIONS OF THE DERIVATIVE

3. \(8x^2 - 10xy + 3y^2 = 26\)

\[
\frac{d}{dx}(8x^2 - 10xy + 3y^2) = \frac{d}{dx}(26)
\]
\[
16x - \frac{d}{dx}(10xy) + \frac{d}{dx}(3y^2) = 0
\]
\[
16x - 10x \frac{dy}{dx} - y \frac{d}{dx}(10x) + 6y \frac{dy}{dx} = 0
\]
\[
16x - 10x \frac{dy}{dx} - 10y + 6y \frac{dy}{dx} = 0
\]
\[
(-10x + 6y) \frac{dy}{dx} = -16x + 10y
\]
\[
\frac{dy}{dx} = \frac{-16x + 10y}{-10x + 6y}
\]
\[
\frac{dy}{dx} = \frac{8x - 5y}{5x - 3y}
\]

5. \(5x^3 = 3y^2 + 4y\)

\[
\frac{d}{dx}(5x^3) = \frac{d}{dx}(3y^2 + 4y)
\]
\[
15x^2 = \frac{d}{dx}(3y^2) + \frac{d}{dx}(4y)
\]
\[
15x^2 = 6y \frac{dy}{dx} + 4 \frac{dy}{dx}
\]
\[
\frac{15x^2}{6y + 4} = \frac{dy}{dx}
\]

6. \(3x^3 - 8y^2 = 10y\)

\[
\frac{d}{dx}(3x^3 - 8y^2) = \frac{d}{dx}(10y)
\]
\[
\frac{d}{dx}(3x^3) - \frac{d}{dx}(8y^2) = 10 \frac{dy}{dx}
\]
\[
9x^2 - 16y \frac{dy}{dx} = 10 \frac{dy}{dx}
\]
\[
9x^2 = (16y + 10) \frac{dy}{dx}
\]
\[
\frac{9x^2}{16y + 10} = \frac{dy}{dx}
\]

7. \(3x^2 = \frac{2 - y}{2 + y}\)

\[
\frac{d}{dx}(3x^2) = \frac{d}{dx} \left( \frac{2 - y}{2 + y} \right)
\]
\[
6x = \left( \frac{2 + y}{(2 + y)^2} \right) \frac{d}{dx}(2 - y) - (2 - y) \frac{d}{dx} \left( \frac{2 + y}{(2 + y)^2} \right)
\]
\[
6x = \left( \frac{2 + y}{(2 + y)^2} \right) \frac{dy}{dx} - (2 - y) \frac{d}{dx} \left( \frac{2 + y}{(2 + y)^2} \right)
\]
\[
6x = \left( \frac{-4 \frac{dy}{dx}}{(2 + y)^2} \right)
\]
\[
6x(2 + y)^2 = -4 \frac{dy}{dx}
\]
\[
-3x(2 + y)^2 = \frac{dy}{dx}
\]

8. \(2y^2 = \frac{5 + x}{5 - x}\)

\[
\frac{d}{dx}(2y^2) = \frac{d}{dx} \left( \frac{5 + x}{5 - x} \right)
\]
\[
4y \frac{dy}{dx} = \frac{(1)(5 - x) - (-1)(5 + x)}{(5 - x)^2}
\]
\[
4y \frac{dy}{dx} = \frac{5 - x + 5 + x}{(5 - x)^2}
\]
\[
4y \frac{dy}{dx} = \frac{10}{(5 - x)^2}
\]
\[
\frac{dy}{dx} = \frac{10}{4y(5 - x)^2}
\]
\[
\frac{dy}{dx} = \frac{5}{2y(5 - x)^2}
\]
9. \(2\sqrt{x} + 4\sqrt[3]{y} = 5y\)

\[
\frac{d}{dx}(2x^{1/2} + 4y^{1/2}) = \frac{d}{dx}(5y)
\]
\[
x^{-1/2} + 2y^{-1/2} \frac{dy}{dx} = 5 \frac{dy}{dx}
\]
\[
(2y^{-1/2} - 5) \frac{dy}{dx} = -x^{-1/2}
\]
\[
\frac{dy}{dx} = \frac{x^{-1/2}}{5 - 2y^{-1/2}} \left(\frac{x^{1/2}y^{1/2}}{x^{1/2}y^{1/2}}\right)
\]
\[
= \frac{y^{1/2}}{x^{1/2}(5y^{1/2} - 2)}
\]
\[
= \frac{\sqrt{y}}{\sqrt{x}(5\sqrt{y} - 2)}
\]

10. \(4\sqrt{x} - 8\sqrt[3]{y} = 6y^{3/2}\)

\[
\frac{d}{dx}(4x^{1/2} - 8y^{1/2}) = 6 \frac{d}{dx}(y^{3/2})
\]
\[
2x^{-1/2} - 4y^{-1/2} \frac{dy}{dx} = 6 \cdot 3 \frac{dy}{dx}
\]
\[
2x^{-1/2} = (9y^{1/2} + 4y^{-1/2}) \frac{dy}{dx}
\]
\[
\frac{2x^{-1/2}}{9y^{1/2} + 4y^{-1/2}} = \frac{dy}{dx}
\]
\[
\frac{2\sqrt{y}}{\sqrt{x}(9y + 4)} = \frac{dy}{dx}
\]

11. \(x^4y^3 + 4x^{3/2} = 6y^{3/2} + 5\)

\[
\frac{d}{dx}(x^4y^3 + 4x^{3/2}) = \frac{d}{dx}(6y^{3/2} + 5)
\]
\[
\frac{d}{dx}(x^4y^3) + \frac{d}{dx}(4x^{3/2}) = \frac{d}{dx}(6y^{3/2}) + \frac{d}{dx}(5)
\]
\[
4x^3y^3 + x^4 \cdot 3y^2 \frac{dy}{dx} + 6x^{1/2} = 9y^{1/2} \frac{dy}{dx} + 0
\]
\[
4x^3y^3 + 6x^{1/2} = 9y^{1/2} \frac{dy}{dx} - 3x^4y^2 \frac{dy}{dx}
\]
\[
4x^3y^3 + 6x^{1/2} = (9y^{1/2} - 3x^4y^2) \frac{dy}{dx}
\]
\[
\frac{4x^3y^3 + 6x^{1/2}}{9y^{1/2} - 3x^4y^2} = \frac{dy}{dx}
\]

12. \((xy)^{1/3} + x^{1/3} = y^{6} + 1\)

\[
\frac{d}{dx}[(xy)^{1/3} + x^{1/3}] = \frac{d}{dx}(y^{6} + 1)
\]
\[
\frac{d}{dx}(x^{4/3}y^{4/3}) + \frac{d}{dx}(x^{1/3}) = \frac{d}{dx}(y^{6}) + \frac{d}{dx}(1)
\]
\[
x^{4/3} \cdot \frac{4}{3} y^{1/3} \frac{dy}{dx} + \frac{4}{3} x^{1/3} y^{4/3} + \frac{1}{3} x^{-2/3}
\]
\[
= 6y^5 \frac{dy}{dx} + 0
\]
\[
\frac{4}{3} x^{1/3} y^{4/3} + \frac{1}{3} x^{-2/3} = 6y^5 \frac{dy}{dx} - \frac{4}{3} x^{4/3} y^{1/3} y^{4/3} \frac{dy}{dx}
\]
\[
4x^{1/3} y^{4/3} + x^{-2/3} = 18y^5 \frac{dy}{dx} - 4x^{1/3} y^{1/3} \frac{dy}{dx}
\]
\[
4x^{1/3} y^{4/3} + x^{-2/3} = (18y^5 - 4x^{4/3} y^{1/3}) \frac{dy}{dx}
\]
\[
\frac{4x^{1/3} y^{4/3} + x^{-2/3}}{18y^5 - 4x^{1/3} y^{1/3}} = \frac{dy}{dx}
\]
\[
\frac{x^{2/3}}{18x^{2/3} y^{5} - 4x^{2} y^{1/3}} = \frac{dy}{dx}
\]

13. \(e^{x^2}y = 5x + 4y + 2\)

\[
\frac{d}{dx}(e^{x^2}y) = \frac{d}{dx}(5x + 4y + 2)
\]
\[
e^{x^2}y \frac{d}{dx}(e^{x^2}y) = \frac{d}{dx}(5x) + \frac{d}{dx}(4y) + \frac{d}{dx}(2)
\]
\[
e^{x^2}y \left(2xy + x^2 \frac{dy}{dx}\right) = 5 + 4 \frac{dy}{dx} + 0
\]
\[
2x y e^{x^2}y \frac{dy}{dx} = 5 + 4 \frac{dy}{dx}
\]
\[
x^2 e^{x^2}y \frac{dy}{dx} - 4 \frac{dy}{dx} = 5 - 2xy e^{x^2}y
\]
\[
(x^2 e^{x^2}y - 4) \frac{dy}{dx} = 5 - 2xy e^{x^2}y
\]
\[
\frac{dy}{dx} = \frac{5 - 2xy e^{x^2}y}{x^2 e^{x^2}y - 4}
\]
14. \( x^2e^y + y = x^3 \)

\[
\frac{d}{dx}(x^2e^y + y) = \frac{d}{dx}(x^3)
\]

\[
\frac{d}{dx}(x^2e^y) + \frac{dy}{dx} = 3xe^y
\]

\[
2xe^y + x^2e^y \frac{dy}{dx} + \frac{dy}{dx} = 3xe^y
\]

\[
x^2e^y \frac{dy}{dx} + \frac{dy}{dx} = 3x^2 - 2xe^y
\]

\[
(x^2e^y + 1) \frac{dy}{dx} = 3x^2 - 2xe^y
\]

\[
\frac{dy}{dx} = \frac{3x^2 - 2xe^y}{x^2e^y + 1}
\]

15. \( x + \ln y = x^2y^3 \)

\[
\frac{d}{dx}(x + \ln y) = \frac{d}{dx}(x^2y^3)
\]

\[
1 + \frac{1}{y} \frac{dy}{dx} = 2xy^3 + 3x^2y^2 \frac{dy}{dx}
\]

\[
\frac{1}{y} \frac{dy}{dx} - 3x^2y^2 \frac{dy}{dx} = 2xy^3 - 1
\]

\[
\left( \frac{1}{y} - 3x^2y^2 \right) \frac{dy}{dx} = 2xy^3 - 1
\]

\[
\frac{dy}{dx} = \frac{2xy^3 - 1}{\frac{1}{y} - 3x^2y^2}
\]

\[
= \frac{y(2xy^3 - 1)}{1 - 3x^2y^3}
\]

16. \( y \ln x + 2 = x^{3/2}y^{5/2} \)

\[
\frac{d}{dx}(y \ln x + 2) = \frac{d}{dx}(x^{3/2}y^{5/2})
\]

\[
\ln x \frac{dy}{dx} + \frac{y}{x} + 0 = \frac{3}{2} x^{1/2}y^{1/2} + \frac{5}{2} x^{3/2}y^{3/2} \frac{dy}{dx}
\]

\[
\ln x \frac{dy}{dx} - \frac{5}{2} x^{3/2}y^{3/2} \frac{dy}{dx} = \frac{3}{2} x^{1/2}y^{1/2} - \frac{y}{x}
\]

\[
\frac{dy}{dx} \left( \ln x - \frac{5}{2} x^{3/2}y^{3/2} \right) = \frac{3}{2} x^{1/2}y^{1/2} - \frac{y}{x}
\]

\[
\frac{dy}{dx} = \frac{\frac{3}{2} x^{1/2}y^{1/2} - \frac{y}{x}}{\ln x - \frac{5}{2} x^{3/2}y^{3/2}}
\]

\[
= \frac{3x^{1/2}y^{1/2} - y}{x(2 \ln x - 5x^{3/2}y^{3/2})}
\]

17. \( x^2 + y^2 = 25; \) tangent at \((-3, 4)\)

\[
\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)
\]

\[
2x + 2y \frac{dy}{dx} = 0
\]

\[
2y \frac{dy}{dx} = -2x
\]

\[
\frac{dy}{dx} = -\frac{x}{y}
\]

\[
m = -\frac{x}{y} = -\frac{-3}{4} = \frac{3}{4}
\]

\[
y - y_1 = m(x - x_1)
\]

\[
y - 4 = \frac{3}{4}(x - (-3))
\]

\[
4y - 16 = 3x + 9
\]

\[
4y = 3x + 25
\]

\[
y = \frac{3}{4}x + 25
\]

18. \( x^2 + y^2 = 100; \) tangent at \((8, -6)\)

\[
\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(100)
\]

\[
2x + 2y \frac{dy}{dx} = 0
\]

\[
2y \frac{dy}{dx} = -\frac{x}{y}
\]

\[
\frac{dy}{dx} = -\frac{x}{y}
\]

\[
m = -\frac{x}{y} = -\frac{8}{-6} = \frac{4}{3}
\]

\[
y - y_1 = m(x - x_1)
\]

\[
y - (-6) = \frac{4}{3}(x - 8)
\]

\[
3y + 18 = 4x - 32
\]

\[
3y = 4x - 50
\]

19. \( x^2y^2 = 1; \) tangent at \((-1, 1)\)

\[
\frac{d}{dx}(x^2y^2) = \frac{d}{dx}(1)
\]

\[
x^2 \frac{d}{dx}(y^2) + y^2 \frac{d}{dx}(x^2) = 0
\]

\[
x^2(2y) \frac{dy}{dx} + y^2(2x) = 0
\]

\[
2x^2y \frac{dy}{dx} = -2xy^2
\]

\[
\frac{dy}{dx} = -\frac{2xy^2}{2x^2y} = -\frac{y}{x}
\]
\[ m = -\frac{y}{x} = -\frac{1}{-1} = 1 \]
\[ y - 1 = 1[x - (-1)] \]
\[ y = x + 1 + 1 \]
\[ y = x + 2 \]

20. \( x^2y^3 = 8 \); tangent at \((-1, 2)\)

\[ \frac{d}{dx}(x^2y^3) = \frac{d}{dx}(8) \]
\[ 2xy^3 + 3x^2y^2 \frac{dy}{dx} = 0 \]
\[ \frac{dy}{dx} = -\frac{2xy^3}{3x^2y^2} \]
\[ m = -\frac{2xy^3}{3x^2y^2} = -\frac{2(-1)(2)^3}{3(-1)^2(2)^2} \]
\[ = \frac{16}{12} = \frac{4}{3} \]
\[ y - 2 = \frac{4}{3}(x + 1) \]
\[ 3y - 6 = 4x + 4 \]
\[ 3y = 4x + 10 \]

21. \( 2y^2 - \sqrt{x} = 4 \); tangent at \((16, 2)\)

\[ \frac{d}{dx}(2y^2 - \sqrt{x}) = \frac{d}{dx}(4) \]
\[ 4y \frac{dy}{dx} - \frac{1}{2}x^{-1/2} = 0 \]
\[ 4y \frac{dy}{dx} = \frac{1}{2x^{1/2}} \]
\[ \frac{dy}{dx} = \frac{1}{8yx^{1/2}} \]
\[ m = \frac{1}{8yx^{1/2}} = \frac{1}{8(2)(16)^{1/2}} \]
\[ = \frac{1}{8(2)(4)} = \frac{1}{64} \]
\[ y - 2 = \frac{1}{64}(x - 16) \]
\[ 64y - 128 = x - 16 \]
\[ 64y = x + 112 \]
\[ y = \frac{x + 7}{64} \]
22. \( y + \sqrt{\frac{x}{y}} = 3 \); tangent at \((4, 2)\)

\[
\begin{align*}
\frac{d}{dx} \left( y + \sqrt{\frac{x}{y}} \right) &= \frac{d}{dx}(3) \\
\frac{dy}{dx} + \frac{d}{dx} \left( \frac{\sqrt{x}}{y} \right) &= 0 \\
\frac{dy}{dx} + \frac{y \left( \frac{1}{2} \right) x^{-1/2} - \sqrt{\frac{x}{y}} \frac{dy}{dx}}{y^2} &= 0 \\
\frac{dy}{dx} &= \frac{-\frac{1}{2} y x^{-1/2} + \sqrt{\frac{x}{y}} \frac{dy}{dx}}{y^2} \\
y^2 \frac{dy}{dx} &= \frac{1}{2} y x^{-1/2} + \sqrt{\frac{x}{y}} \frac{dy}{dx} \\
(y^2 - \sqrt{x}) \frac{dy}{dx} &= -\frac{1}{2} y x^{-1/2} \\
\frac{dy}{dx} &= \frac{-y}{2x^{1/2}(y^2 - \sqrt{x})} \\
m &= \frac{-y}{2x^{1/2}(y^2 - \sqrt{x})} \\
&= \frac{-2}{2(2)(4 - 2)} \\
&= \frac{-1}{4} \\
y - 2 &= \frac{-1}{4} (x - 4) \\
y &= \frac{1}{4} x + 3 \\
x + 4y &= 12
\end{align*}
\]

23. \( e^{x^2+y^2} = xe^{5y} - y^2 e^{5x/2} \); tangent at \((2, 1)\)

\[
\begin{align*}
\frac{d}{dx} (e^{x^2+y^2}) &= \frac{d}{dx} (xe^{5y} - y^2 e^{5x/2}) \\
\frac{d}{dx} (x^2 + y^2) &= e^{5y} + x \frac{d}{dx} (e^{5y}) - \left[ 2y \frac{dy}{dx} e^{5x/2} + y^2 e^{5x/2} \frac{d}{dx} \left( \frac{5x}{2} \right) \right] \\
e^{x^2+y^2} \left( 2x + 2y \frac{dy}{dx} \right) &= e^{5y} + x \cdot 5e^{5y} \frac{dy}{dx} - 2ye^{5x/2} \frac{dy}{dx} - \frac{5}{2} y^2 e^{5x/2} \\
(2ye^{x^2+y^2} - 5xe^{5y} + 2ye^{5x/2}) \frac{dy}{dx} &= -2xe^{x^2+y^2} + e^{5y} - \frac{5}{2} y^2 e^{5x/2} \\
\frac{dy}{dx} &= \frac{-2xe^{x^2+y^2} + e^{5y} - \frac{5}{2} y^2 e^{5x/2}}{2ye^{x^2+y^2} - 5xe^{5y} + 2ye^{5x/2}} \\
m &= \frac{-4e^5 + e^5 - \frac{5}{2} e^5}{2e^5 - 10e^5 + 2e^5} = \frac{-\frac{11}{2} e^5}{-6e^5} = \frac{11}{12} \\
y - 1 &= \frac{11}{12} (x - 2) \\
y &= \frac{11}{12} x - \frac{5}{6}
\end{align*}
\]
24. $2xe^{xy} = e^{x^2} + ye^{x^2}$; tangent at $(1,1)$

$$\frac{d}{dx}(2xe^{xy}) = \frac{d}{dx}(e^{x^2} + ye^{x^2})$$

$$2e^{xy} + 2x \cdot e^{xy} \frac{d}{dx}(xy) = e^{x^2} \frac{d}{dx}(x^2) + \frac{dy}{dx} \cdot e^{x^2} + ye^{x^2} \cdot \frac{d}{dx}(x^2)$$

$$2xe^{xy} + 2xe^{xy} \left(y + x \frac{dy}{dx}\right) = e^{x^2} \cdot 3x^2 + \frac{dy}{dx} \cdot e^{x^2} + ye^{x^2} \cdot 2x$$

$$(2x^2e^{xy} - e^{x^2}) \frac{dy}{dx} = -2e^{xy} - 2xye^{xy} + 2xye^{x^2} + 3x^2e^{x^3}$$

$$\frac{dy}{dx} = \frac{-2e^{xy} - 2xye^{xy} + 2xye^{x^2} + 3x^2e^{x^3}}{2x^2e^{xy} - e^{x^2}}$$

$$m = \frac{-2e - 2e + 2e + 3e}{2e - e} = \frac{e}{e} = 1$$

$$y - 1 = 1(x - 1)$$

$$y = x$$

25. $\ln(x + y) = x^3y^2 + \ln(x^2 + 2) - 4$; tangent at $(1,2)$

$$\frac{d}{dx}[\ln(x + y)] = \frac{d}{dx}[x^3y^2 + \ln(x^2 + 2) - 4]$$

$$\frac{1}{x + y} \cdot \frac{d}{dx}(x + y) = 3x^2y^2 + x^3 \cdot 2y \frac{dy}{dx} + \frac{1}{x^2 + 2} \frac{d}{dx}(x^2 + 2) - \frac{d}{dx}(4)$$

$$\left(\frac{1}{x + y} - 2x^3y\right) \frac{dy}{dx} = 3x^2y^2 + \frac{2x}{x^2 + 2} - \frac{1}{x + y}$$

$$\frac{dy}{dx} = \frac{3x^2y^2 + \frac{2x}{x^2 + 2} - \frac{1}{x + y}}{\frac{1}{x + y} - 2x^3y}$$

$$m = \frac{3 \cdot 1 \cdot 4 + \frac{2 \cdot 4}{1} - \frac{1}{3}}{\frac{1}{3} - 2 \cdot 1 \cdot 2} = \frac{\frac{12}{3} - \frac{1}{3}}{\frac{1}{3}} = -\frac{37}{11}$$

$$y - 2 = -\frac{37}{11}(x - 1)$$

$$y = -\frac{37}{11}x + \frac{59}{11}$$

26. $\ln(x^2 + y^2) = \ln(5x) + \frac{y}{x} - 2$; tangent at $(1,2)$

$$\frac{d}{dx}[\ln(x^2 + y^2)] = \frac{d}{dx}[\ln(5x) + \frac{y}{x} - 2]$$

$$\frac{1}{x^2 + y^2} \cdot \frac{d}{dx}(x^2 + y^2) = \frac{1}{5x} \cdot \frac{d}{dx}(5x) + \frac{x \frac{dy}{dx} - y}{x^2} - \frac{d}{dx}(2)$$

$$\frac{1}{x^2 + y^2} \left(2x + 2y \frac{dy}{dx}\right) = \frac{1}{x} + \frac{1}{x} \cdot \frac{dy}{dx} - \frac{y}{x^2}$$

$$\left(\frac{2y}{x^2 + y^2} - \frac{1}{x}\right) \frac{dy}{dx} = \frac{1}{x} \cdot \frac{y}{x^2} - \frac{2x}{x^2 + y^2}$$

$$\frac{dy}{dx} = \frac{1}{x} \cdot \frac{y}{x^2} - \frac{2x}{x^2 + y^2} - \frac{1}{x}$$
412  Chapter 6  APPLICATIONS OF THE DERIVATIVE

\[
m = \frac{1 - 2 - \frac{2}{5}}{\frac{4}{5} - 1} = \frac{-\frac{3}{5}}{-\frac{1}{5}} = 7
\]

\[
y - 2 = 7(x - 1) \\
y = 7x - 5
\]

27. \(y^3 + xy - y = 8x^4; \ x = 1\)

First, find the \(y\)-value of the point.

\[y^3 + (1)y - y = 8(1)^4 \]
\[y^3 = 8 \]
\[y = 2\]

The point is \((1, 2)\).

Find \(\frac{dy}{dx}\)

\[3y^2 \frac{dy}{dx} + x \frac{dy}{dx} + y - \frac{dy}{dx} = 32x^3 \]
\[(3y^2 + x - 1) \frac{dy}{dx} = 32x^3 - y \]
\[\frac{dy}{dx} = \frac{32x^3 - y}{3y^2 + x - 1}\]

At \((1, 2)\),

\[\frac{dy}{dx} = \frac{32(1)^3 - 2}{3(2)^2 + 1 - 1} = \frac{30}{12} = \frac{5}{2}\]
\[y - 2 = \frac{5}{2}(x - 1) \]
\[y - 2 = \frac{5}{2}x - \frac{5}{2} \]
\[y = \frac{5}{2}x - \frac{1}{2}\]

28. \(y^3 + 2x^2y - 8y = x^3 + 19, \ x = 2\)

\[3y^2 \frac{dy}{dx} + 2x^2 \frac{dy}{dx} + 4xy - 8 \frac{dy}{dx} = 3x^2 \]
\[(3y^2 + 2x^2 - 8) \frac{dy}{dx} = 3x^2 - 4xy \]
\[\frac{dy}{dx} = \frac{3x^2 - 4xy}{3y^2 + 2x^2 - 8}\]

Find \(y\) when \(x = 2\).

\[y^3 + 8y - 8y = 8 + 19 \]
\[y^3 = 27 \]
\[y = 3\]

\[\frac{dy}{dx} = \frac{12 - 24}{27 + 8 - 8} = \frac{-12}{27} = -\frac{4}{9}\]
\[y - 3 = -\frac{4}{9}(x - 2) \]
\[y - 3 = -\frac{4}{9}x + \frac{8}{9} \]
\[y = -\frac{4}{9}x + \frac{35}{9}\]
29. \( y^3 + xy^2 + 1 = x + 2y^2; \ x = 2 \)

Find the \( y \)-value of the point.

\[
\begin{align*}
3y^2 + 2y^2 + 1 & = 2 + 2y^2 \\
y^3 + 1 & = 2 \\
y^3 & = 1 \\
y & = 1
\end{align*}
\]

The point is \((2, 1)\).

Find \( \frac{dy}{dx} \).

\[
\begin{align*}
3y^2 \frac{dy}{dx} + x 2y \frac{dy}{dx} + y^2 & = 1 + 4y \frac{dy}{dx} \\
3y^2 \frac{dy}{dx} + 2xy \frac{dy}{dx} - 4y \frac{dy}{dx} & = 1 - y^2 \\
(3y^2 + 2xy - 4y) \frac{dy}{dx} & = 1 - y^2
\end{align*}
\]

\[
\frac{dy}{dx} = \frac{1 - y^2}{3y^2 + 2xy - 4y}
\]

At \((2, 1)\),

\[
\frac{dy}{dx} = \frac{1 - 1^2}{3(1)^2 + 2(2)(1) - 4(1)} = 0.
\]

\[
y - 0 = 0(x - 2) \\
y = 1
\]

30. \( y^4(1 - x) + xy = 2, \ x = 1 \)

Find the \( y \)-value.

\[
\begin{align*}
y^4(1 - 1) + y & = 2 \\
y & = 2
\end{align*}
\]

The point is \((1, 2)\).

Find \( \frac{dy}{dx} \).

\[
\begin{align*}
y^4(-1) + 4y^3(1 - x) \frac{dy}{dx} + x \frac{dy}{dx} + y & = 0 \\
- y^4 + 4y^3 \frac{dy}{dx} - 4xy^3 \frac{dy}{dx} + x \frac{dy}{dx} + y & = 0
\end{align*}
\]

\[
(4y^3 - 4xy^3 + x) \frac{dy}{dx} = y^4 - y
\]

\[
\frac{dy}{dx} = \frac{y^4 - y}{4y^3 - 4xy^3 + x}
\]

\( \frac{dy}{dx} \) at \((1, 2)\) is

\[
\frac{2^4 - 2}{4 \cdot 2^3 - 4(1)^2 + 1} = \frac{16 - 2}{32 - 32 + 1} = 14.
\]

\[
y - 2 = 14(x - 1) \\
y - 2 = 14x - 14 \\
y = 14x - 12
\]
31. $2y^3(x - 3) + x\sqrt{y} = 3; \ x = 3$

Find the $y$-value of the point.

$$2y^3(3 - 3) + 3\sqrt{y} = 3$$
$$3\sqrt{y} = 3$$
$$\sqrt{y} = 1$$
$$y = 1$$

The point is $(3, 1)$

Find $\frac{dy}{dx}$.

$$2y^3(1) + 6y^2(x - 3) \frac{dy}{dx} + x \left( \frac{1}{2} \right) y^{-1/2} \frac{dy}{dx} + \sqrt{y} = 0$$

$$6y^2(x - 3) \frac{dy}{dx} + \frac{x \frac{dy}{dx}}{2\sqrt{y}} = -2y^3 - \sqrt{y}$$

$$\left[ 6y^2(x - 3) + \frac{x}{2\sqrt{y}} \right] \frac{dy}{dx} = -2y^3 - \sqrt{y}$$

$$\frac{dy}{dx} = \frac{-2y^3 - \sqrt{y}}{6y^2(x - 3) + \frac{x}{2\sqrt{y}}}$$

$$= \frac{-4y^{7/2} - 2y}{12y^{5/2}(x - 3) + x}$$

At $(3, 1)$,

$$\frac{dy}{dx} = \frac{-4(1) - 2}{12(1)(3 - 3) + 3} = \frac{-6}{3} = -2.$$

$$y - 1 = -2(x - 3)$$
$$y - 1 = -2x + 6$$
$$y = -2x + 7$$

32. $\frac{y}{18}(x^2 - 64) + x^{2/3}y^{1/3} = 12; \ x = 8$

Find the $y$-value of the point.

$$\frac{y}{18}(64 - 64) + 8^{2/3}y^{1/3} = 12$$

$$4y^{1/3} = 12$$
$$y^{1/3} = 3$$
$$y = 27$$

The point is $(8, 27)$.

Find $\frac{dy}{dx}$.

$$\frac{y}{18}(2x) + \frac{1}{18}(x^2 - 64) \frac{dy}{dx} + \frac{1}{3}x^{2/3}y^{-2/3} \frac{dy}{dx}$$

$$+ \frac{2}{3}x^{-1/3}y^{1/3} = 0$$
\[ \frac{dy}{dx} = \frac{-2x\frac{2}{3}y^{-\frac{2}{3}}}{\frac{18}{3} + \frac{x^{2/3}y^{-2/3}}{3}} \]

\[ \frac{dy}{dx} = \frac{-2xy - 2x^{-1/3}y^{1/3}}{18 + x^{2/3}y^{-2/3}} \]

\[ \frac{dy}{dx} = \frac{-2xy - 12x^{-1/3}y^{1/3}}{x^2 - 64 + 6x^{2/3}y^{-2/3}} \]

\[ \frac{dy}{dx} \text{ at } (8, 27) \text{ is} \]

\[ \frac{-2(8)(27) - 12(8)^{-1/3}(27)^{1/3}}{64 - 64 + 6(8)^{2/3}(27)^{-2/3}} = \frac{-432 - 18}{2^3} \]

\[ = \frac{-450}{2^3} \left( \frac{9}{24} \right) \]

\[ = \frac{-675}{4} \]

\[ y - 27 = \frac{-675}{4}(x - 8) \]

\[ y - 27 = \frac{-675}{4}x + 1350 \]

\[ y = \frac{-675}{4}x + 1377 \]

33. \( x^2 + y^2 = 100 \)

(a) Lines are tangent at points where \( x = 6 \). By substituting \( x = 6 \) in the equation, we find that the points are \( (6, 8) \) and \( (6, -8) \).

\[ \frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(100) \]

\[ 2x + 2y \frac{dy}{dx} = 0 \]

\[ 2y \frac{dy}{dx} = -2x \]

\[ dy = \frac{-x}{y} \]

\[ m_1 = \frac{x}{y} = \frac{6}{8} = \frac{3}{4} \]

\[ m_2 = \frac{x}{y} = \frac{6}{-8} = \frac{3}{4} \]

First tangent:

\[ y - 8 = \frac{3}{4}(x - 6) \]

\[ y = \frac{3}{4}x + \frac{25}{2} \]

Second tangent:

\[ y - (-8) = \frac{3}{4}(x - 6) \]

\[ y + 8 = \frac{3}{4}x - \frac{18}{4} \]

\[ y = \frac{3}{4}x - \frac{25}{2} \]
(b)

34. $x^{2/3} + y^{2/3} = 2$; (1, 1)

Find $\frac{dy}{dx}$

$$\frac{2}{3} x^{-1/3} + \frac{2}{3} y^{-1/3} \frac{dy}{dx} = 0$$
$$\frac{2}{3} y^{-1/3} \frac{dy}{dx} = -\frac{2}{3} x^{-1/3}$$
$$\frac{dy}{dx} = -\frac{2}{3} x^{-1/3} \frac{y^{-1/3}}{x^{1/3}}$$
$$\frac{dy}{dx} = -\frac{y^{1/3}}{x^{1/3}}$$

At (1, 1)

$$\frac{dy}{dx} = -\frac{1^{1/3}}{1^{1/3}} = -1$$
$$y - 1 = -1(x - 1)$$
$$y - 1 = -x + 1$$
$$y = -x + 2$$

35. $3(x^2 + y^2)^2 = 25(x^2 - y^2)$; (2, 1)

Find $\frac{dy}{dx}$

$$6(x^2 + y^2) \frac{d}{dx}(x^2 + y^2) = 25 \frac{d}{dx}(x^2 - y^2)$$
$$6(x^2 + y^2) \left(2x + 2y \frac{dy}{dx}\right) = 25 \left(2x - 2y \frac{dy}{dx}\right)$$
$$12x^3 + 12x^2 y \frac{dy}{dx} + 12xy^2 + 12y^3 \frac{dy}{dx} = 50x - 50y \frac{dy}{dx}$$
$$12x^2 y \frac{dy}{dx} + 12y^3 \frac{dy}{dx} + 50y \frac{dy}{dx} = -12x^3 - 12xy^2 + 50x$$
$$12x^2 y + 12y^3 + 50y \frac{dy}{dx} = -12x^3 - 12xy^2 + 50x$$
$$\frac{dy}{dx} = \frac{-12x^3 - 12xy^2 + 50x}{12x^2 y + 12y^3 + 50y}$$
At $(2, 1)$,

\[
\frac{dy}{dx} = \frac{-12(2)^3 - 12(2)(1)^2 + 50(2)}{12(2)^2 + 12(1)^3 + 50(1)}
\]

\[
= \frac{-20}{110} = \frac{-2}{11}
\]

\[
y - 1 = -\frac{2}{11}(x - 2)
\]

\[
y - 1 = -\frac{2}{11}x + \frac{4}{11}
\]

\[
y = -\frac{2}{11}x + \frac{15}{11}
\]

**36.** $y^2(x^2 + y^2) = 20x^2$; $(1, 2)$

Find $\frac{dy}{dx}$.

\[
2y(x^2 + y^2) \frac{dy}{dx} + y^2 \left(2x + 2y \frac{dy}{dx}\right) = 40x
\]

\[
2x^2y \frac{dy}{dx} + 2y^3 \frac{dy}{dx} + 2xy^2 + 2y^3 \frac{dy}{dx} = 40x
\]

\[
2x^2y \frac{dy}{dx} + 4y^3 \frac{dy}{dx} = -2xy^2 + 40x
\]

\[
(2x^2y + 4y^3) \left(\frac{dy}{dx}\right) = -2xy^2 + 40x
\]

\[
\frac{dy}{dx} = \frac{-2xy^2 + 40x}{2x^2y + 4y^3}
\]

At $(1, 2)$,

\[
\frac{dy}{dx} = \frac{-2(1)(2)^2 + 40(1)}{2(1)^2(2) + 4(2)^3}
\]

\[
= \frac{32}{36} = \frac{8}{9}
\]

\[
y - 2 = \frac{8}{9}(x - 1)
\]

\[
y - 2 = \frac{8}{9}x - \frac{8}{9}
\]

\[
y = \frac{8}{9}x + \frac{10}{9}
\]
37. \(2(x^2 + y^2)^2 = 25xy^2; \ (2, 1)\)

Find \(\frac{dy}{dx}\)

\[
4(x^2 + y^2) \frac{d}{dx}(x^2 + y^2) = 25 \frac{d}{dx}(xy^2)
\]

\[
4(x^2 + y^2) \left(2x + 2y \frac{dy}{dx}\right) = 25 \left(y^2 + 2xy \frac{dy}{dx}\right)
\]

\[
8x^3 + 8x^2y \frac{dy}{dx} + 8xy^2 + 8y^3 \frac{dy}{dx} = 25y^2 + 50xy \frac{dy}{dx}
\]

\[
8x^2y \frac{dy}{dx} + 8y^3 \frac{dy}{dx} - 50xy \frac{dy}{dx} = -8x^3 - 8xy^2 + 25y^2
\]

\[
(8x^2y + 8y^3 - 50xy) \left(\frac{dy}{dx}\right) = -8x^3 - 8xy^2 + 25y^2
\]

\[
\frac{dy}{dx} = \frac{-8x^3 - 8xy^2 + 25y^2}{8x^2y + 8y^3 - 50xy}
\]

At \((2, 1)\),

\[
\frac{dy}{dx} = \frac{-8(2)^3 - 8(2)(1)^2 + 25(1)^2}{8(2)^4(1) + 8(1)^3 - 50(2)(1)}
\]

\[
= \frac{-55}{-60} = \frac{11}{12}
\]

\[
y - 1 = \frac{11}{12}(x - 2)
\]

\[
y - 1 = \frac{11}{12}x - \frac{11}{6}
\]

\[
y = \frac{11}{12}x - \frac{5}{6}
\]

38. \(x^2 + y^2 + 1 = 0\)

\[
\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(-1)
\]

\[
2x + 2y \frac{dy}{dx} = 0
\]

\[
\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}
\]

If \(x\) and \(y\) are real numbers, \(x^2\) and \(y^2\) are nonnegative. 1 plus a nonnegative number cannot equal zero, so there is no function \(y = f(x)\) that satisfies \(x^2 + y^2 + 1 = 0\).

39. \(y^2 = x^3 + ax + b\)

\[
\frac{d}{dx}(y^2) = \frac{d}{dx}(x^3 + ax + b)
\]

\[
2y \frac{dy}{dx} = 3x^2 + a
\]

\[
\frac{dy}{dx} = \frac{3x^2 + a}{2y}
\]
40. \( \sqrt{x} + \sqrt{2v+1} = 5 \)

\[
\frac{du}{dv}(\sqrt{x} + \sqrt{2v+1}) = \frac{du}{dv} (5)
\]

\[
\frac{1}{2} u^{-1/2} \frac{du}{dv} + \frac{1}{2} (2v+1)^{-1/2} (2) = 0
\]

\[
\frac{1}{2} u^{-1/2} \frac{du}{dv} = -\frac{1}{(2v+1)^{1/2}}
\]

\[
\frac{du}{dv} = -\frac{2u^{1/2}}{(2v+1)^{1/2}}
\]

41. \( \sqrt{x} + \sqrt{2v+1} = 5 \)

\[
\frac{dv}{du}(\sqrt{x} + \sqrt{2v+1}) = \frac{dv}{du} (5)
\]

\[
\frac{1}{2} u^{-1/2} + \frac{1}{2} (2v+1)^{-1/2} (2) \frac{dv}{du} = 0
\]

\[
(2v+1)^{-1/2} \frac{dv}{du} = -\frac{1}{2} u^{-1/2}
\]

\[
\frac{dv}{du} = -(2v+1)^{1/2}
\]

42. \( 2p^2 + q^2 = 1600 \)

(a) \( 4p + 2q \frac{dq}{dp} = 0 \)

\[
4p = -2q \frac{dq}{dp}
\]

\[
-\frac{2p}{q} = \frac{dq}{dp}
\]

This is the rate of change of demand with respect to price.

(b) \( 4p \frac{dp}{dq} + 2q = 0 \)

\[
\frac{dp}{dq} = -\frac{q}{2p}
\]

This is the rate of change of price with respect to demand.

43. \( C^2 = x^2 + 100\sqrt{x} + 50 \)

(a) \( 2C \frac{dC}{dx} = 2x + \frac{1}{2} (100)x^{-1/2} \)

\[
\frac{dC}{dx} = \frac{2x + 50x^{-1/2}}{2C}
\]

\[
\frac{dC}{dx} = \frac{x + 25x^{-1/2}}{C} \cdot x^{1/2}
\]

\[
\frac{dC}{dx} = \frac{x^{3/2} + 25}{Cx^{1/2}}
\]

When \( x = 5 \), the approximate increase in cost of an additional unit is

\[
\frac{(5)^{3/2} + 25}{(5^2 + 100\sqrt{5} + 50)^{1/2}(5)^{1/2}} = \frac{36.18}{(17.28)\sqrt{5}}
\]

\[
\approx 0.94.
\]

(b) \( 900(x-5)^2 + 25R^2 = 22,500 \)

\[
R^2 = 900 - 36(x - 5)^2
\]

\[
2R \frac{dR}{dx} = -72(x - 5)
\]

\[
\frac{dR}{dx} = \frac{-36(x - 5)}{R} = \frac{180 - 36x}{R}
\]

When \( x = 5 \), the approximate change in revenue for a unit increase in sales is

\[
\frac{180 - 36(5)}{R} = 0
\]

44. First note that

if \( \log R(w) = 1.83 - 0.43 \log(w) \)

then \( R(w) = 10^{1.83-0.43\log(w)} \)

\[
= 10^{1.8310^{-0.43\log(w)}}
\]

\[
= 10^{1.83(10^{\log(w)})^{-0.43}}
\]

\[
= 10^{1.83w^{-0.43}}
\]

(a) \( \frac{d}{dw} \log R(w) = \frac{d}{dw} [1.83 - 0.43 \log(w)] \)

\[
\frac{1}{\ln 10} \frac{1}{R(w)} \frac{dR}{dw} = 0 - 0.43 \frac{1}{\ln 10} w
\]

\[
\frac{dR}{dw} = -0.43 \frac{R(w)}{w}
\]

\[
\approx -0.43 \frac{10^{1.83}w^{-0.43}}{w}
\]

\[
\approx -29.0716w^{-1.43}
\]

(b) \( R(w) = 10^{1.83w^{-0.43}} \)

\[
\frac{d}{dw} R(w) = \frac{d}{dw} [10^{1.83}w^{-0.43}] \]

\[
\frac{dR}{dw} = 10^{1.83}(-0.43)w^{-1.43}
\]

\[
\approx -29.0716w^{-1.43}
\]
45. \( b - a = (b + a)^3 \)
\[
\frac{d}{db}(b-a) = \frac{d}{db}((b+a)^3) \\
1 - \frac{da}{db} = 3(b+a)^2 \frac{d}{db}(b+a) \\
1 - \frac{da}{db} = 3(b+a)^2 \left(1 + \frac{da}{db}\right) \\
1 - \frac{da}{db} = 3(b+a)^2 + 3(b+a)^2 \frac{da}{db} \\
-\frac{da}{db} - 3(b+a)^2 \frac{da}{db} = 3(b+a)^2 - 1 \\
[-1 - 3(b+a)^2] \frac{da}{db} = 3(b+a)^2 - 1 \\
\frac{da}{db} = \frac{3(b+a)^2 - 1}{-1 - 3(b+a)^2} \\
\frac{da}{db} = 0 \\
3(b+a)^2 - 1 = 0 \\
\frac{b+a}{3} = \frac{1}{\sqrt[3]{3}} \\
b + a = \frac{1}{\sqrt[3]{3}}
\]
Since \( b - a = (b + a)^3 \) \( \Rightarrow (\frac{1}{\sqrt[3]{3}})^3 = \frac{1}{3\sqrt[3]{3}} \).
\[
b + a = \frac{1}{\sqrt[3]{3}} \\
-(b-a) = -\frac{1}{3\sqrt[3]{3}} \\
2a = \frac{2}{3\sqrt[3]{3}} \\
a = \frac{1}{3\sqrt[3]{3}}
\]

46.
\[
xy^a = k \\
\frac{d}{dx}(xy^a) = \frac{d}{dx}(k) \\
x \frac{d}{dx}(y^a) + y^a(1) = 0 \\
x \left( ay^{a-1} \frac{dy}{dx} \right) + y^a = 0 \\
a xy^{a-1} \frac{dy}{dx} = -y^a \\
\frac{dy}{dx} = -\frac{y^a}{a xy^{a-1}} \\
\frac{dy}{dx} = -\frac{y}{ax}
\]

47. \( s^3 - 4st + 2t^3 - 5t = 0 \)
\[
3s^2 \frac{ds}{dt} - \left(4t \frac{ds}{dt} + 4s\right) + 6t^2 - 5 = 0 \\
3s^2 \frac{ds}{dt} - 4t \frac{ds}{dt} - 4s + 6t^2 - 5 = 0 \\
\frac{ds}{dt}(3s^2 - 4t) = 4s - 6t^2 + 5 \\
\frac{ds}{dt} = \frac{4s - 6t^2 + 5}{3s^2 - 4t}
\]

48. \( 2s^2 + \sqrt{st} - 4 = 3t \)
\[
4s \frac{ds}{dt} + \frac{1}{2}(st)^{-1/2} \left(s + \frac{ds}{dt}\right) = 3 \\
4s \frac{ds}{dt} + s + t \frac{ds}{dt} = 3 \\
\frac{8s(\sqrt{st})}{2\sqrt{st}} \frac{ds}{dt} + s + t \frac{ds}{dt} = 3 \\
\frac{(8s\sqrt{st} + t)}{2\sqrt{st}} \frac{ds}{dt} + s = 3 \\
\frac{(8s\sqrt{st} + t)}{2\sqrt{st}} \frac{ds}{dt} = 6\sqrt{st} - s \\
\frac{ds}{dt} = -s + 6\sqrt{st} \frac{1}{8s\sqrt{st} + t}
\]

6.5 Related Rates

1. \( y^2 - 8x^3 = -55; \frac{dx}{dt} = -4, x = 2, y = 3 \)
\[
2y \frac{dy}{dt} - 24x^2 \frac{dx}{dt} = 0 \\
\frac{dy}{dt} = 12x^2 \frac{dx}{dt} \\
\frac{dy}{dt} = 48(-4) \\
\frac{dy}{dt} = -64
\]

2. \( 8y^3 + x^2 = 1; \frac{dx}{dt} = 2, x = 3, y = -1 \)
\[
24y^2 \frac{dy}{dt} + 2x \frac{dx}{dt} = 0 \\
\frac{dy}{dt} = -\frac{2x \frac{dx}{dt}}{24y^2} = -\frac{x \frac{dx}{dt}}{12y^2} \\
= \frac{(3)(2)}{12(-1)^2} \\
= \frac{1}{2}
\]
3. \(2xy - 5x + 3y^3 = -51; \ \frac{dx}{dt} = -6, x = 3, y = -2\)

\[
2x \frac{dy}{dt} + 2y \frac{dx}{dt} - 5 \frac{dx}{dt} + 9y^2 \frac{dy}{dt} = 0
\]

\[
(2x + 9y^2) \frac{dy}{dt} + (2y - 5) \frac{dx}{dt} = 0
\]

\[
(2x + 9y^2) \frac{dy}{dt} = (5 - 2y) \frac{dx}{dt}
\]

\[
\frac{dy}{dt} = \frac{5 - 2y}{2x + 9y^2} \frac{dx}{dt}
\]

\[
= \frac{5 - 2(-2)}{2(3) + 9(-2)^2} \cdot (-6)
\]

\[
= \frac{9}{-42} \cdot (-6) = \frac{-54}{42} = \frac{-9}{7}
\]

4. \(4x^3 - 6xy^2 + 3y^2 = 228; \ \frac{dx}{dt} = 3, x = -3, y = 4\)

\[
12x^2 \frac{dx}{dt} - 6y^2 \frac{dx}{dt} - 12xy \frac{dy}{dt} + 6y \frac{dy}{dt} = 0
\]

\[
(6y - 12xy) \frac{dy}{dt} = (6y^2 - 12x^2) \frac{dx}{dt}
\]

\[
\frac{dy}{dt} = \frac{y^2 - 2x^2}{y - 2xy} \frac{dx}{dt}
\]

\[
= \frac{4^2 - 2 \cdot (-3)^2}{4 \cdot 2 \cdot (-3)} \cdot 4
\]

\[
= \frac{-6}{28} \cdot 3 = \frac{-18}{28} = \frac{-9}{14}
\]

5. \(\frac{x^2 + y}{x - y} = 9; \ \frac{dx}{dt} = 2, x = 4, y = 2\)

\[
(x - y) \left(2x \frac{dx}{dt} + 2y \frac{dy}{dt}\right) - (x^2 + y) \left(\frac{dx}{dt} - \frac{dy}{dt}\right) = 0
\]

\[
2(x - y)^2 \frac{dx}{dt} + (x - y) \frac{dx}{dt} - (x^2 + y) \frac{dx}{dt} + (x^2 + y) \frac{dy}{dt} = 0
\]

\[
[2(x - y) - (x^2 + y)] \frac{dx}{dt} + [(x - y) + (x^2 + y)] \frac{dy}{dt} = 0
\]

\[
\frac{dy}{dt} = \frac{[(x^2 + y) - 2x(x - y)]}{(x - y) + (x^2 + y)} \frac{dx}{dt}
\]

\[
= \frac{(-x^2 + y + 2x^2y)}{x + x^2} \frac{dx}{dt}
\]

\[
= \frac{[-(4)^2 + 2 + 2(4)(2)]}{4 + 4^2}
\]

\[
= \frac{4}{20} = \frac{1}{5}
\]

6. \(\frac{y^3 - 4x^2}{x^3 + 2y} = \frac{44}{31}; \ \frac{dx}{dt} = 5, x = -3, y = -2\)

\[
31(y^3 - 4x^2) = 44(x^3 + 2y)
\]

\[
31y^3 - 124x^2 = 44x^3 + 88y
\]

\[
93y^2 \frac{dy}{dt} - 248x \frac{dx}{dt} = 132x^2 \frac{dx}{dt} + 88 \frac{dy}{dt}
\]

\[
(93y^2 - 88) \frac{dy}{dt} = (132x^2 + 248x) \frac{dx}{dt}
\]

\[
\frac{dy}{dt} = \frac{132x^2 + 248x}{93y^2 - 88} \frac{dx}{dt}
\]

\[
= \frac{132(-3)^2 + 248(-3)}{93(-3)^2 - 88} \cdot 5
\]

\[
= \frac{444}{284} \cdot 5 = \frac{2220}{284} = \frac{555}{71}
\]

7. \(xe^y = 3 + \ln x; \ \frac{dx}{dt} = 6, x = 2, y = 0\)

\[
e^y \frac{dx}{dt} + xe^y \frac{dy}{dt} = 0 + 1 \frac{dx}{x}
\]

\[
e^y \frac{dy}{dt} = \left(\frac{1}{x} - e^y\right) \frac{dx}{dt}
\]

\[
\frac{dy}{dt} = \frac{(1 - e^y) \frac{dx}{dt}}{xe^y}
\]

\[
= \frac{(1 - (2)e^0)(6)}{2e^0}
\]

\[
= \frac{-6}{4} = -\frac{3}{2}
\]

8. \(y \ln x + xe^y = 1; \ \frac{dx}{dt} = 5, x = 1, y = 0\)

\[
\frac{d}{dt}(y \ln x) + \frac{d}{dt}(xe^y) = \frac{d}{dt}(1)
\]

\[
\ln x \frac{dy}{dt} + y \frac{dx}{dt} + e^y \frac{dy}{dt} + x \cdot e^y \frac{dy}{dt} = 0
\]

\[
(\ln x + xe^y) \frac{dy}{dt} + \left(\frac{y}{x} + e^y\right) \frac{dx}{dt} = 0
\]

\[
(\ln x + xe^y) \frac{dy}{dt} = -\left(\frac{y}{x} + e^y\right) \frac{dx}{dt}
\]

\[
\frac{dy}{dt} = \frac{\left(\frac{y}{x} + e^y\right) \frac{dx}{dt}}{\ln x + xe^y}
\]

\[
= -\frac{(y + xe^y) \frac{dx}{dt}}{\ln x + xe^y}
\]

\[
= -\frac{[0 + (1)e^0](5)}{(1)\ln 1 + 1^2e^0} = -5
\]
9. \( C = 0.2x^2 + 10,000; \ x = 80 \ \frac{dx}{dt} = 12 \)

\[
\frac{dC}{dt} = 0.2(2x) \frac{dx}{dt} = 0.2(160)(12) = 384
\]

The cost is changing at a rate of $384 per month.

10. \( C = \frac{R^2}{450,000} + 12,000; \ \frac{dC}{dx} = 15 \)

\[
R = 25,000
\]

\[
\frac{dC}{dx} = \frac{R}{225,000} \frac{dR}{dx}
\]

\[
15 = \frac{25,000}{225,000} \frac{dR}{dx}
\]

\[
135 = \frac{dR}{dx}
\]

Revenue is changing at a rate of $135 per unit.

11. \( R = 50x - 0.4x^2; \ C = 5x + 15; \ x = 40; \ \frac{dx}{dt} = 10 \)

(a) \[
\frac{dR}{dt} = 50 \frac{dx}{dt} - 0.8x \frac{dx}{dt}
\]

\[
= 50(10) - 0.8(40)(10)
\]

\[
= 500 - 320 = 180
\]

Revenue is increasing at a rate of $180 per day.

(b) \[
\frac{dC}{dt} = 5 \frac{dx}{dt} = 5(10) = 50
\]

Cost is increasing at a rate of $50 per day.

(c) Profit = Revenue - Cost

\[
P = R - C
\]

\[
\frac{dP}{dt} = \frac{dR}{dt} - \frac{dC}{dt} = 180 - 50 = 130
\]

Profit is increasing at a rate of $130 per day.

12. \( R = 50x - 0.4x^2; \ C = 5x + 15; \ x = 80; \ \frac{dx}{dt} = 12 \)

(a) \[
\frac{dR}{dt} = 50 \frac{dx}{dt} - 0.8x \frac{dx}{dt}
\]

\[
= 50(12) - 0.8(80)(12)
\]

\[
= 600 - 768 = -168
\]

Revenue is decreasing at a rate of $168 per day.

(b) \[
\frac{dC}{dt} = 5 \frac{dx}{dt} = 5(12) = 60
\]

Cost is increasing at a rate of $60 per day.

13. \( pq = 8000; \ p = 3.50, \ \frac{dp}{dt} = 0.15 \)

\[
pq = 8000
\]

\[
p \frac{dq}{dt} + q \frac{dp}{dt} = 0
\]

\[
\frac{dq}{dt} = -\frac{q \frac{dp}{dt}}{p}
\]

\[
= -\left(\frac{8000}{3.50}\right)(0.15)
\]

\[
\approx -98
\]

Demand is decreasing at a rate of approximately 98 units per unit time.

14. \( R = pq; \ \frac{dq}{dt} = 25 \)

Find the relationship between \( p \) and \( q \) by finding the equation of the line through \((0, 70)\), and \((100, 60)\).

\[
m = \frac{70 - 60}{6 - 100} = \frac{-10}{-100} = \frac{1}{10}
\]

\[
p - 70 = -\frac{1}{10}(q - 0)
\]

\[
p - 70 = -\frac{1}{10}q
\]

\[
p = -\frac{1}{10}q + 70
\]

\[
R = \left(-\frac{1}{10}q + 70\right)q
\]

\[
= -\frac{1}{10}q^2 + 70q
\]

\[
\frac{dR}{dt} = -\frac{1}{5} \frac{dq}{dt} + 70 \frac{dq}{dt}
\]

\[
= -\frac{1}{5}(20)(25) + 70(25)
\]

\[
= -100 + 1750 = 1650
\]

Revenue is increasing at a rate of $1650 per day.
15. $V = k(R^2 - r^2)$; $k = 555.6$, $R = 0.02$ mm, $\frac{dR}{dt} = 0.003$ mm per minute; $r$ is constant.

$$V = k(R^2 - r^2)$$
$$V = 555.6(R^2 - r^2)$$
$$\frac{dV}{dt} = 555.6 \left( 2R \frac{dR}{dt} - 0 \right)$$
$$= 555.6(0.02)(0.003) = 0.067 \text{ mm/min}$$

16. $y = nx^m$

Note that $n$ is a constant.

$$\ln y = \ln (nx^m)$$
$$\ln y = \ln n + \ln x^m$$
$$\ln y = \ln n + m \ln x$$

Now take the derivative of both sides with respect to $t$.

$$\frac{1}{y} \frac{dy}{dt} = 0 + m \frac{1}{x} \frac{dx}{dt}$$
$$\frac{1}{y} \frac{dy}{dt} = m \frac{1}{x} \frac{dx}{dt}$$

17. $b = 0.22m^{0.87}$

$$\frac{db}{dt} = 0.22(0.87)m^{0.13} \frac{dm}{dt}$$
$$= 0.1914m^{0.13} \frac{dm}{dt}$$
$$\frac{dm}{dt} = \frac{m^{0.13}}{0.1914} \frac{db}{dt}$$
$$= \frac{25^{0.13}}{0.1914}$$
$$= 1.9849$$

The rate of change of the total weight is about $1.9849$ g/day.

18. $E = 429m^{-0.35}$

$$\frac{dE}{dt} = 429(-0.35)m^{-1.35} \frac{dm}{dt}$$
$$= -150.15m^{-1.35} \frac{dm}{dt}$$
$$= -150.15(0.25)(0.001)$$
$$= -0.0067$$

The rate of change of the energy expenditure is about $-0.0067$ cal/g/hr$^2$.

19. $r = 140.2m^{0.75}$

(a) $\frac{dr}{dt} = 140.2(0.75)m^{-0.25} \frac{dm}{dt}$
$$= 105.15m^{-0.25} \frac{dm}{dt}$$

(b) $\frac{dr}{dt} = 105.15(250)^{-0.25}(2)$
$$\approx 52.89$$

The rate of change of the average daily metabolic rate is about $52.89$ kcal/day$^2$.

20. $E = 26.5w^{-0.34}$

$$\frac{dE}{dt} = 26.5(-0.34)w^{-1.34} \frac{dw}{dt}$$
$$= -9.01w^{-1.34} \frac{dw}{dt}$$
$$= -9.01(5)^{-1.34}(0.05)$$
$$\approx -0.0521$$

The rate of change of the energy expenditure is about $-0.0521$ kcal/kg/km/day.

21. $C = \frac{1}{10}(T - 60)^2 + 100$

$$\frac{dC}{dt} = \frac{1}{5}(T - 60) \frac{dT}{dt}$$

If $T = 76^\circ$ and $\frac{dT}{dt} = 8$,

$$\frac{dC}{dt} = \frac{1}{5}(76 - 60)(8) = \frac{1}{5}(16)(8)$$
$$= 25.6.$$}

The crime rate is rising at the rate of 25.6 crimes/month.

22. $W(t) = \frac{-0.02t^2 + t}{t + 1}$

$$\frac{dW}{dt} = \frac{(-0.04t + 1)(t + 1) - (1)(-0.02t^2 + t)}{(t + 1)^2}$$

If $t = 5$,

$$\frac{dW}{dt} = \frac{(-0.2 + 1)(6) - (-0.5 + 5)}{6^2}$$
$$= \frac{4.8 - 4.5}{36}$$
$$= 0.008.$$
23. Let \( x \) = the distance of the base of the ladder from the base of the building; 
\( y \) = the distance up the side of the building to the top of the ladder.

Find \( \frac{dy}{dt} \) when \( x = 8 \) ft and \( \frac{dx}{dt} = 9 \) ft/min.

Since \( y = \sqrt{17^2 - x^2} \), when \( x = 8 \), 
\( y = 15 \).

By the Pythagorean theorem, 
\[
2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0
\]
\[
\frac{dy}{dt} = -\frac{2x}{2y} \cdot \frac{dx}{dt} = -\frac{x}{y} \frac{dx}{dt}
\]
\[
= -\frac{8}{15} (9) = -\frac{24}{5}
\]

The ladder is sliding down the building at the rate of \( \frac{24}{5} \) ft/min.

24. (a) Let \( x \) = the distance one car travels west; 
\( y \) = the distance the other car travels north; 
\( s \) = the distance between the two cars.

Use \( d = rt \) to find \( x \) and \( y \).
\[
(40) \frac{dx}{dt} = 80 \text{ mi}
\]
\[
(30) \frac{dy}{dt} = 60 \text{ mi}
\]
\[
s = \sqrt{x^2 + y^2} = \sqrt{(80)^2 + (60)^2} = 100
\]

The distance between the cars after 2 hours is 100 mi.
\[
\frac{dx}{dt} = 40 \text{ mph and } \frac{dy}{dt} = 30 \text{ mph}
\]
\[
\frac{ds}{dt} = \frac{(40)(40) + (60)(30)}{100} = 50
\]

The distance between the two cars is changing at the rate of 50 mph.

(b) From part (a), we have 
\[
\frac{ds}{dt} = x \frac{dx}{dt} + y \frac{dy}{dt}
\]

Use \( d = rt \) to find \( x \) and \( y \). When the second car has traveled 1 hour, the first car has traveled 2 hours.
\[
(40) \frac{dx}{dt} = 40 \text{ mi}
\]
\[
(30) \frac{dy}{dt} = 60 \text{ mi}
\]
\[
s = \sqrt{x^2 + y^2} = \sqrt{(40)^2 + (60)^2} = 72.11
\]

The distance between the cars after the second car has traveled 1 hour is about 72.11 mi.
\[
\frac{ds}{dt} = \frac{(40)(40) + (60)(30)}{72.11} \approx 47.15
\]

The distance between the two cars is changing at a rate of about 47.15 mph.
25. Let \( r \) = the radius of the circle formed by the ripple.

Find \( \frac{dA}{dt} \) when \( r = 4 \) ft and \( \frac{dr}{dt} = 2 \) ft/min.

\[
A = \pi r^2 \\
\frac{dA}{dt} = 2\pi r \frac{dr}{dt} \\
= 2\pi(4)(2) \\
= 16\pi
\]

The area is changing at the rate of 16\( \pi \) ft\(^2\)/min.

26. \( V = \frac{1}{3}\pi r^3 \), \( r = 4 \) in, and \( \frac{dr}{dt} = -\frac{1}{4} \) in/hr

\[
\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \\
= 4\pi(4)^2 \left( -\frac{1}{4} \right) \\
= -16\pi \text{ in}^3/\text{hr}
\]

27. \( V = x^3 \), \( x = 3 \) cm, and \( \frac{dV}{dx} = 2 \) cm\(^3\)/min

\[
\frac{dV}{dt} = 3x^2 \frac{dx}{dt} \\
\frac{dx}{dt} = \frac{1}{3} \frac{dV}{dt} \\
= \frac{1}{3 \cdot 3^2} (2) \\
= \frac{2}{27} \text{ cm/min}
\]

28. Let \( r \) = the radius of the base of the conical pile.

Find \( \frac{dV}{dt} \) when \( r = 6 \) in, and \( \frac{dr}{dt} = 0.75 \) in/min.

\( h = 2r \) for all \( t \).

\[
V = \frac{\pi}{3} r^2 h \\
V = \frac{\pi}{3} r^2 (2r) \\
= \frac{2\pi}{3} r^3 \\
\frac{dV}{dt} = \frac{3 \cdot 2\pi r^2}{3} \frac{dr}{dt} \\
\frac{dV}{dt} = 2\pi(6^2)(0.75) \\
= 54\pi
\]

The volume is changing at the rate of 54\( \pi \) in\(^3\)/min.

29. Let \( y \) = the length of the man’s shadow; \( x \) = the distance of the man from the lamp post; \( h \) = the height of the lamp post.

\[
\frac{dx}{dt} = 50 \text{ ft/min}
\]

Find \( \frac{dy}{dt} \) when \( x = 25 \) ft.

Now \( \frac{h}{x+y} = \frac{6}{y} \), by similar triangles. When \( x = 8, \ y = 10 \),

\[
\frac{10.8}{x+y} = \frac{6}{y} \\
10.8y = 6x + 6y \\
4.8y = 6x \\
y = 1.25x \\
\frac{dy}{dt} = 1.25 \frac{dx}{dt} \\
= 1.25(50) \\
\frac{dy}{dt} = 62.5
\]

The length of the shadow is increasing at the rate of 62.5 ft/min.
30. Let \( x = \) one-half the width of the triangular cross section; \( h = \) the height of the water; \( V = \) the volume of the water.

\[
\frac{dV}{dt} = 4 \text{ cu ft per min}
\]

Find \( \frac{dh}{dt} \) when \( h = 4 \).

\[
V = \left( \frac{\text{Area of triangular cross section}}{\text{length}} \right) \cdot (\text{length})
\]

Area of triangular cross section

\[
= \frac{1}{2} (\text{base})(\text{height})
= \frac{1}{2} (2x)(h) = xh
\]

By similar triangles,

\[
\frac{6}{2x} = \frac{6}{h} \quad \text{so} \quad x = \frac{h}{2}.
\]

\[
V = (xh)(16) = \left( \frac{h}{2} \right) 16 = 8h^2
\]

\[
\frac{dV}{dt} = 16h \frac{dh}{dt}
\]

\[
\frac{1}{16h} \frac{dV}{dt} = \frac{dh}{dt}
\]

\[
\frac{1}{16(4)} (4) = \frac{dh}{dt}
\]

\[
\frac{dh}{dt} = \frac{1}{16}
\]

The height of the water is increasing at a rate of \( \frac{1}{16} \) ft/min.

31. Let \( x = \) the distance from the docks \( s = \) the length of the rope.

\[
\frac{ds}{dt} = 1 \text{ ft/sec}
\]

\[
s^2 = x^2 + (8)^2
\]

\[
2s \frac{ds}{dt} = 2x \frac{dx}{dt} + 0
\]

\[
s \frac{ds}{dt} = x \frac{dx}{dt}
\]

If \( x = 8 \),

\[
s = \sqrt{(8)^2 + (8)^2} = \sqrt{128} = 8\sqrt{2}.
\]

Then,

\[
8\sqrt{2}(1) = 8 \frac{dx}{dt}
\]

\[
\frac{dx}{dt} = \sqrt{2} \approx 1.41
\]

The boat is approaching the deck at \( \sqrt{2} \approx 1.41 \) ft/sec.

32. Let \( x = \) the horizontal length; \( r = \) the rope length.
By the Pythagorean theorem,
\[ x^2 + 100^2 = 200^2 \]
\[ x = \sqrt{30000} = 100\sqrt{3}. \]
\[ r^2 = x^2 + 100^2 \]
\[ 2r \frac{dr}{dt} = 2x \frac{dx}{dt} + 0 \]
\[ r \frac{dr}{dt} = \frac{dx}{dt} \]
\[ \frac{dr}{dt} = \frac{x \frac{dx}{dt}}{r} \]
\[ \frac{dr}{dt} = \frac{100\sqrt{3}(50)}{200} = 25\sqrt{3} \approx 43.3 \]

She must let out the string at a rate of \(25\sqrt{3} \approx 43.3\) ft/min.

### 6.6 Differentials: Linear Approximation

1. \( y = 2x^3 - 5x; \ x = -2, \Delta x = 0.1 \)
   \[ dy = (6x^2 - 5) \, dx \]
   \[ \Delta y \approx (6x^2 - 5) \Delta x \approx [6(-2)^2 - 5](0.1) \approx 1.9 \]

2. \( y = 4x^3 - 3x; \ x = 3, \Delta x = 0.2 \)
   \[ dy = (12x^2 - 3) \, dx \]
   \[ \Delta y \approx (12x^2 - 3) \Delta x \approx [12(3)^2 - 3](0.2) \approx 21 \]

3. \( y = x^3 - 2x^2 + 3, \ x = 1, \Delta x = -0.1 \)
   \[ dy = (3x^2 - 4x) \, dx \]
   \[ \Delta y \approx (3x^2 - 4x) \Delta x \]
   \[ = [3(1)^2 - 4(1)](-0.1) \]
   \[ = 0.1 \]

4. \( y = 2x^3 + x^2 - 4x; \ x = 2, \Delta x = -0.2 \)
   \[ dy = (6x^2 + 2x - 4) \, dx \]
   \[ \approx (6x^2 + 2x - 4) \Delta x \]
   \[ = [6(2)^2 + 2(2) - 4](-0.2) \]
   \[ = (24 + 4 - 4)(-0.2) = 24 \]

5. \( y = \sqrt{3x + 2}, \ x = 4, \Delta x = 0.15 \)
   \[ dy = 3 \left( \frac{1}{2} (3x + 2)^{-1/2} \right) \, dx \]
   \[ \Delta y \approx \frac{3}{2\sqrt{3x + 2}} \Delta x \approx \frac{3}{2(3.74)}(0.15) \approx 0.060 \]

6. \( y = \sqrt{4x - 1}; \ x = 5, \Delta x = 0.08 \)
   \[ dy = \frac{1}{2}(4x - 1)^{-1/2} 4 \, dx \]
   \[ = 2(4x - 1)^{-1/2} 4 \, dx \]
   \[ \approx 2(4x - 1)^{-1/2} \Delta x \]
   \[ = 2[4(5) - 1]^{-1/2}(0.08) \]
   \[ = 2(19)^{-1/2}(0.08) \]
   \[ = 2(0.08) \]
   \[ = 0.037 \]

7. \( y = \frac{2x - 5}{x + 1}; \ x = 2, \Delta x = -0.03 \)
   \[ dy = \frac{(x + 1)(2) - (2x - 5)(1)}{(x + 1)^2} \, dx \]
   \[ = \frac{7}{(x + 1)^2} \, dx \]
   \[ = \frac{7}{(2 + 1)^2} \Delta x \]
   \[ = \frac{7}{(2 + 1)^2}(-0.03) \]
   \[ = -0.023 \]

8. \( y = \frac{6x - 3}{2x + 1}; \ x = 3, \Delta x = -0.04 \)
   \[ dy = \frac{6(2x + 1) - 2(6x - 3)}{(2x + 1)^2} \, dx \]
   \[ = \frac{12}{(2x + 1)^2} \, dx \]
   \[ = \frac{12}{(2 + 1)^2} \Delta x \]
   \[ = \frac{12}{(2 + 1)^2}(-0.04) \]
   \[ = -0.48 \]
   \[ = \frac{49}{49} \]

9. √145
   We know \(\sqrt{144} = 12\), so \(f(x) = \sqrt{x}, x = 144, dx = 1\).
   \[ dy = \frac{1}{2}x^{-1/2} \]
   \[ = \frac{1}{2\sqrt{x}} \, dx \]
   \[ = \frac{1}{2\sqrt{144}} \, dx \]
   \[ = \frac{1}{24}(1) = \frac{1}{24} \]
   \[ \sqrt{145} \approx f(x) + dy = 12 + \frac{1}{24} \]
   \[ \approx 12.0417 \]

\[ \sqrt{145} \approx f(x) + dy = 12 + \frac{1}{24} \]
   \[ \approx 12.0417 \]
By calculator, \( \sqrt{145} \approx 12.0416 \).
The difference is \( |12.0417 - 12.0416| = 0.0001 \).

10. \( \sqrt{23} \)

We know \( \sqrt{25} = 5 \), \( f(x) = \sqrt{x} \), \( x = 25 \), and 
\( dx = -2 \).
\[
\frac{dy}{dx} = \frac{1}{2} x^{-1/2}
\]
\[
dy = \frac{1}{2\sqrt{x}} dx = \frac{1}{2\sqrt{25}}(-2) = -\frac{1}{5} = -0.2.
\]
\( \sqrt{23} \approx f(x) + dy = 5 - 0.2 = 4.8 \)

By calculator, \( \sqrt{23} \approx 4.7958 \).
The difference is \( |4.8 - 4.7958| = 0.0042 \).

11. \( \sqrt{0.99} \)

We know \( \sqrt{1} = 1 \), so \( f(x) = \sqrt{x} \), \( x = 1 \), \( dx = -0.01 \).
\[
\frac{dy}{dx} = \frac{1}{2} x^{-1/2}
\]
\[
dy = \frac{1}{2\sqrt{x}} dx = \frac{1}{2\sqrt{1}}(-0.01) = -0.005
\]
\( \sqrt{0.99} \approx f(x) + dy = 1 - 0.005 = 0.995 \)

By calculator, \( \sqrt{0.99} \approx 0.9950 \).
The difference is \( |0.995 - 0.9950| = 0 \).

12. \( \sqrt{17.02} \)

We know \( \sqrt{16} = 4 \), \( f(x) = \sqrt{x} \), \( x = 16 \), and 
\( dx = 1.02 \).
\[
\frac{dy}{dx} = \frac{1}{2} x^{-1/2}
\]
\[
dy = \frac{1}{2\sqrt{x}} dx = \frac{1}{2\sqrt{16}}(1.02) = \frac{1}{8}(1.02) = 0.1275
\]
\( \sqrt{17.02} \approx f(x) + dy = 4 + 0.1275 = 4.1275 \)

By calculator, \( \sqrt{17.02} \approx 4.1255 \).
The difference is \( |4.1275 - 4.1255| = 0.0020 \).

13. \( e^{0.01} \)

We know \( e^0 = 1 \), so \( f(x) = e^x \), \( x = 0 \), \( dx = 0.01 \).
\[
\frac{dy}{dx} = e^x
\]
\[
dy = e^x dx
\]
\[
dy = e^0(0.01) = 0.01
\]
\( e^{0.01} \approx f(x) + dy = 1 + 0.01 = 1.01 \)

By calculator, \( e^{0.01} \approx 1.0101 \).
The difference is \( |1.01 - 1.0101| = 0.0001 \).

14. \( e^{-0.002} \)

We know \( e^0 = 1 \), \( f(x) = e^x \), \( x = 0 \), and 
\( dx = -0.002 \).
\[
\frac{dy}{dx} = e^x
\]
\[
dy = e^x dx = e^0(-0.002) = -0.002
\]
\( e^{-0.002} \approx f(x) + dy = 1 - 0.002 = 0.998 \)

By calculator, \( e^{-0.002} \approx 0.9980 \).
The difference is \( |0.9980 - 0.998| = 0 \).

15. \( \ln 1.05 \)

We know \( \ln 1 = 0 \), so \( f(x) = \ln x \), \( x = 1 \), \( dx = 0.05 \).
\[
\frac{dy}{dx} = \frac{1}{x}
\]
\[
dy = \frac{1}{x} dx
\]
\[
dy = \frac{1}{1}(0.05) = 0.05
\]
\( \ln 1.05 \approx f(x) + dy = 0 + 0.05 = 0.05 \)

By calculator, \( \ln 1.05 \approx 0.0488 \).
The difference is \( |0.05 - 0.0488| = 0.0012 \).

16. \( \ln 0.98 \)

We know \( \ln 1 = 0 \), \( f(x) = \ln x \), \( x = 1 \), and 
\( dx = -0.02 \).
\[
\frac{dy}{dx} = \frac{1}{x}
\]
\[
dy = \frac{1}{x} dx = \frac{1}{1}(-0.02) = -0.02
\]
\( \ln 0.98 \approx f(x) + dy = 0 - 0.02 = -0.02 \)

By calculator, \( \ln 0.98 \approx -0.0202 \).
The difference is \( |-0.02 - (-0.0202)| = 0.0002 \).
17. Let $D = \text{the demand in thousands of pounds};$
$x = \text{the price in dollars}.$
$D(q) = -3q^3 - 2q^2 + 1500$

(a) $q = 2, \Delta q = 0.10$
$dD = (-9q^2 - 4q) dq$
$\Delta D \approx (-9q^2 - 4q) \Delta q$
$\approx [-9(4) - 4(2)](0.10)$
$\approx -4.4 \text{ thousand pounds}$

(b) $q = 6, \Delta q = 0.15$
$\Delta D \approx [-9(36) - 4(6)](0.15)$
$\approx -52.2 \text{ thousand pounds}$

18. $A(x) = 0.04x^3 + 0.1x^2 + 0.5x + 6$

(a) $x = 3, \Delta x = 1$
$dA = (0.12x^2 + 0.2x + 0.5) dx$
$= (0.12x^2 + 0.2x + 0.5) \Delta x$
$= [(0.12)(3)^2 + (0.2)(3) + (0.5)](1)$
$= 2.18$

(b) $x = 5, \Delta x = 1$
$dA = [(0.12)(5)^2 + (0.2)(5) + (0.5)](1)$
$= 4.5$

19. $R(x) = 12,000 \ln(0.01x + 1)$
$x = 100, \Delta x = 1$
$dR = \frac{12,000}{0.01x + 1} (0.01) dx$
$\Delta R \approx \frac{120}{0.01x + 1} \Delta x$
$\approx \frac{120}{0.01(100) + 1} (1)$
$\approx $60

20. $P = R - C$
$= 12,000 \ln(0.01x + 1) - (150 + 75x)$
$x = 100, \Delta x = 1$
$dP = \frac{12,000}{0.01x + 1} (0.01) dx - 75 dx$
$\Delta P \approx \frac{12,000}{0.01x + 1} (0.01) \Delta x - 75 \Delta x$
$\approx \frac{12,000}{0.01(100) + 1} (0.01)(1) - 75(1)$
$\approx 60 - 75$
$\approx -15$

The change in profit is a loss of about $15.

21. If a cube is given a coating 0.1 in. thick, each edge increases in length by twice that amount, or 0.2 in. because there is a face at both ends of the edge.

$V = x^3, \ x = 4, \ \Delta x = 0.2$
$dV = 3x^2 \ dx$
$\Delta V \approx 3x^2 \Delta x$
$= 3(4^2)(0.2)$
$= 9.6$

For 1000 cubes $9.6(1000) = 9600 \text{ in.}^3$ of coating should be ordered.

22. Let $x = \text{the number of beach balls};$
$V = \text{the volume of } x \text{ beach balls}.$

Then $\frac{dV}{dr} \approx \text{the volume of material in } x \text{ beach balls since they are hollow}.$
$V = \frac{4}{3} \pi r^3 x$
$r = 6 \text{ in.}, \ x = 5000, \ \Delta r = 0.03 \text{ in.}$
$dV = \frac{4}{3} \pi (3r^2 x + r^3) \Delta r$
$= \frac{4}{3} \pi (3 \cdot 36 \cdot 5000 + 216)(0.03)$
$= 21,608.64 \pi$

$21,608 \pi \text{ in.}^3$ of material would be needed.

23. (a) $A(x) = y = 0.003631x^3 - 0.03746x^2 + 0.1012x + 0.009$

Let $x = 1, \ dx = 0.2.$
$\frac{dy}{dx} = 0.010893x^2 - 0.07492x + 0.1012$
$dy = (0.010893x^2 - 0.07492x + 0.1012) dx$
$\Delta y \approx (0.010893x^2 - 0.07492x + 0.1012) \Delta x$
$\approx (0.010893 \cdot 1^2 - 0.07492 \cdot 1 + 0.1012) \cdot 0.2$
$\approx 0.007435$

The alcohol concentration increases by about 0.74 percent.

(b) $\Delta y \approx (0.010893 \cdot 3^2 - 0.07492 \cdot 3 + 0.1012) \cdot 0.2$
$\approx -0.005105$

The alcohol concentration decreases by about 0.51 percent.
24. \( C = \frac{5x}{9 + x^2} \)
\[ dC = \frac{5(9 + x^2) - 2x(5x)}{(9 + x^2)^2} \, dx = \frac{45 + 5x^2 - 10x^2}{(9 + x^2)^2} \, dx = \frac{45 - 5x^2}{(9 + x^2)^2} \, dx \]
\[ \approx \frac{45 - 5x^2}{(9 + x^2)^2} \Delta x \]
(a) \( x = 1, \ \Delta x = 0.5 \)
\[ dC \approx \frac{45 - 5(1)^2}{(9 + 1)^2} (0.5) = \frac{40}{100} (0.5) = 0.2 \]
(b) \( x = 2, \ \Delta x = 0.25 \)
\[ dC \approx \frac{45 - 5(2)^2}{(9 + 4)^2} (0.25) = 0.037 \]

25. \( P(x) = \frac{25x}{8 + x^2} \)
\[ dP = \frac{(8 + x^2)(25) - 25x(2x)}{(8 + x^2)^2} \, dx = \frac{(8 + x^2)(25) - 25x(2x)}{(8 + x^2)^2} \Delta x \]
(a) \( x = 2, \ \Delta x = 0.5 \)
\[ dP = \frac{[(8 + 4)(25) - (25)(2)(4)](0.5)}{(8 + 4)^2} = \frac{347}{2} \text{ million} \]
(b) \( x = 3, \ \Delta x = 0.25 \)
\[ dP = \frac{[(8 + 9)(25) - (25)(3)(4)](0.25)}{(8 + 9)^2} \approx -0.022 \text{ million} \]

26. \( A = \pi r^2, \ r = 1.7 \text{ mm}, \ \Delta r = -0.1 \text{ mm} \)
\[ dA = 2\pi r \, dr \]
\[ \Delta A \approx 2\pi r \Delta r \]
\[ = 2\pi (1.7)(-0.1) = -0.34\pi \text{ mm}^2 \]

27. \( r \) changes from 14 mm to 16 mm, so \( \Delta r = 2 \).
\[ V = \frac{4}{3} \pi r^3 \]
\[ dV = \frac{4}{3} (3) \pi r^2 \, dr \]
\[ \Delta V \approx 4\pi r^2 \Delta r \]
\[ = 4\pi (14)^2 (2) = 1568\pi \text{ mm}^3 \]

28. \( A = \pi r^2, \ r = 1.2 \text{ mi}, \ \Delta r = 0.2 \text{ mi} \)
\[ dA = 2\pi r \, dr \]
\[ \Delta A \approx 2\pi r \Delta r \]
\[ = 2\pi (1.2)(0.2) = 0.48\pi \text{ mi}^2 \]

29. \( r \) increases from 20 mm to 22 mm, so \( \Delta r = 2 \).
\[ A = \pi r^2 \]
\[ dA = 2\pi r \, dr \]
\[ \Delta A \approx 2\pi r \Delta r \]
\[ = 2\pi (20)(2) = 80\pi \text{ mm}^2 \]

30. \( A(p) = \frac{1.181p}{94.359 - p} \)
(a) Since values for \( p \) must be non-negative and the denominator can’t be zero, a sensible domain would be from 0 to about 94.
(b) \[ dA = \frac{(94.359 - p)(1.181) - 1.181p(-1)}{(94.359 - p)^2} \, dp = \frac{111.437979 - 1.181p + 1.181p}{(94.359 - p)^2} \, dp \]
\[ = \frac{111.437979}{(94.359 - p)^2} \, dp \]
We are given \( p = 60 \) and \( dp = 65 - 60 = 5 \).
\[ dA \approx \frac{111.437979}{(94.359 - 60)^2} (5) \approx 0.472 \]
It will take about 0.47 years.
The actual value is
\[ A(65) - A(60) \approx 2.615 - 2.062 = 0.553 \]
or about 0.55 years.
31. \( W(t) = -3.5 + 197.5e^{-0.01394(t-108.4)} \)

(a) \[ dW = 197.5e^{-0.01394(t-108.4)} (-1)e^{-0.01394(t-108.4)} (-0.01394)dt \]
\[ = 2.75315e^{-0.01394(t-108.4)} e^{-0.01394(t-108.4)} dt \]

We are given \( t = 80 \) and \( dt = 90 - 80 = 10 \).

\[ dW \approx 9.258 \]

The pig will gain about 9.3 kg.

(b) The actual weight gain is calculated as
\[ W(90) - W(80) \approx 50.736 - 41.202 = 9.534 \]

or about 9.5 kg.

32. \( V = \frac{4}{3}\pi r^3, \ r = 4 \text{ cm}, \ \Delta r = 0.2 \text{ cm} \)

\[ dV = 4\pi r^2 dr \]
\[ \Delta V \approx 4\pi r^2 \Delta r \]
\[ = 4\pi(4)^2(0.2) \]
\[ = 12.8\pi \text{ cm}^3 \]

33. \( r = 3 \text{ cm}, \ \Delta r = -0.2 \text{ cm} \)

\[ V = \frac{4}{3}\pi r^3 \]
\[ dV = 4\pi r^2 dr \]
\[ \Delta V \approx 4\pi r^2 \Delta r \]
\[ = 4\pi(9)(-0.2) \]
\[ = -7.2\pi \text{ cm}^3 \]

34. \( V = x^3, \ V = 27, \ x = 3, \ \Delta V = 0.1 \)

\[ dV = 3x^2 dx \]
\[ \Delta V \approx 3x^2 \Delta x \]
\[ \Delta x \approx \frac{\Delta V}{3x^2} \]
\[ \approx \frac{0.1}{3 \cdot 3^2} \]
\[ \approx 0.0037 \text{ mm} \]

35. \( V = \frac{1}{3}\pi r^2 h; \ h = 13, \ dh = 0.2 \)

\[ V = \frac{1}{3}\pi \left( \frac{h}{15} \right)^2 h \]
\[ = \frac{\pi}{775} h^3 \]
\[ dV = \frac{\pi}{775} \cdot 3h^2 dh \]
\[ = \frac{\pi}{225} h^2 dh \]
\[ \Delta V \approx \frac{\pi}{225} h^2 \Delta h \]
\[ \approx \frac{\pi}{225} (13^2)(0.2) \]
\[ \approx 0.472 \text{ cm}^3 \]
36. \( A = x^2; \ x = 3.45, \ \Delta x = \pm 0.002 \)
\[ dA = 2x \, dx \]
\[ \Delta A \approx 2x \Delta x = 2(3.45)(\pm 0.002) = \pm 0.0138 \text{ in.}^2 \]

37. \( A = x^2; \ x = 4, dA = 0.01 \)
\[ dA = 2x \, dx \]
\[ \Delta A \approx 2x \Delta x \approx \frac{\Delta A}{2x} \approx \frac{0.01}{2(4)} \approx 0.00125 \text{ cm} \]

38. \( r = 4.87 \text{ in.}, \ \Delta r = \pm 0.040 \)
\[ A = \pi r^2 \]
\[ dA = 2\pi r \, dr \]
\[ \Delta A \approx 2\pi r \Delta r = 2\pi(4.87)(\pm 0.040) = \pm 1.224 \text{ in.}^2 \]

39. \( V = \frac{4}{3} \pi r^3; \ r = 5.81, \ \Delta r = \pm 0.003 \)
\[ dV = \frac{4}{3} \pi (3r^2) \, dr \]
\[ \Delta V \approx \frac{4}{3} \pi (3r^2) \Delta r \]
\[ = 4\pi(5.81)^2(\pm 0.003) \]
\[ = \pm 0.405\pi \approx \pm 1.273 \text{ in.}^3 \]

40. \( V = x^3; \ x = 5, dV = 0.3 \)
\[ dV = 3x^2 \, dx \]
\[ \Delta V \approx 3x^2 \Delta x \]
\[ \Delta x \approx \frac{\Delta V}{3x^2} \approx \frac{0.3}{3(5^2)} \approx 0.004 \text{ ft} \]

41. \( h = 7.284 \text{ in.}, \ r = 1.09 \pm 0.007 \text{ in.} \)
\[ V = \frac{1}{3} \pi r^2 h \]
\[ dV = \frac{2}{3} \pi rh \, dr \]
\[ \Delta V \approx \frac{2}{3} \pi rh \Delta r \]
\[ = \frac{2}{3} \pi (1.09)(7.284)(0.007) \]
\[ = \pm 0.116 \text{ in.}^3 \]

Chapter 6 Review Exercises

1. \( f(x) = -x^3 + 6x^2 + 1; \ [-1, 6] \)
\[ f'(x) = -3x^2 + 12x = 0 \text{ when } x = 0, 4. \]
\[ f(-1) = 8 \]
\[ f(0) = 1 \]
\[ f(4) = 33 \]
\[ f(6) = 1 \]
Absolute maximum of 33 at 4; absolute minimum of 1 at 0 and 6.

2. \( f(x) = 4x^3 - 9x^2 - 3; \ [-1, 2] \)
\[ f'(x) = 12x^2 - 18x = 0 \text{ when } x = 0, \frac{3}{2}. \]
\[ f(-1) = -16 \]
\[ f(0) = -3 \]
\[ f \left( \frac{3}{2} \right) = -9.75 \]
\[ f(2) = -7 \]
Absolute maximum of -3 at 0; absolute minimum of -16 at -1.

3. \( f(x) = x^3 + 2x^2 - 15x + 3; \ [-4, 2] \)
\[ f'(x) = 3x^2 + 4x - 15 = 0 \text{ when } (3x - 5)(x + 3) = 0 \]
\[ x = \frac{5}{3} \text{ or } x = -3. \]
\[ f(-4) = 31 \]
\[ f(-3) = 39 \]
\[ f \left( \frac{5}{3} \right) = -\frac{319}{27} \]
\[ f(2) = -11 \]
Absolute maximum of 39 at -3; absolute minimum of -\frac{319}{27} at \frac{5}{3}.

4. \( f(x) = -2x^3 - 2x^2 + 2x - 1; \ [-3, 1] \)
\[ f'(x) = -6x^2 - 4x + 2 \]
\[ f'(x) = 0 \text{ when } 3x^2 + 2x - 1 = 0 \]
\[ (3x - 1)(x + 1) = 0 \]
\[ x = \frac{1}{3} \text{ or } x = -1. \]
\[ f(-3) = 29 \]
\[ f(-1) = -3 \]
\[ f \left( \frac{1}{3} \right) = -\frac{17}{27} \]
\[ f(1) = -3 \]
Absolute maximum of 29 at -3; absolute minimum of -3 at -1 and 1.

7. (a) \( f(x) = \frac{2\ln x}{x^2} \); \([1, 4]\)

\[
\frac{d}{dx} f(x) = \left( \frac{3}{2} - (2\ln x) \right) \frac{x}{x^4} = \frac{2x - 4x\ln x}{x^4} = \frac{2 - 4\ln x}{x^4}.
\]

\( f'(x) = 0 \) when

\[
2 - 4\ln x = 0 \\
2 = 4\ln x \\
0.5 = \ln x \\
e^{0.5} = x \\
x \approx 1.6487.
\]

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( e^{0.5} )</td>
<td>0.36788</td>
</tr>
<tr>
<td>4</td>
<td>0.17329</td>
</tr>
</tbody>
</table>

Maximum is 0.37; minimum is 0.

(b) \([2, 5]\)

Note that the critical number of \( f \) is not in the domain, so we only test the endpoints.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.34657</td>
</tr>
<tr>
<td>5</td>
<td>0.12876</td>
</tr>
</tbody>
</table>

Maximum is 0.35, minimum is 0.13.

10. \( x^2y^3 + 4xy = 2 \)

\[
d \frac{d}{dx} (x^2y^3 + 4xy) = \frac{d}{dx} (2)
\]

\[
2xy^3 + 3y^2 \left( \frac{dy}{dx} \right) x^2 + 4y + 4x \frac{dy}{dx} = 0 \\
(3x^2y^2 + 4x) \frac{dy}{dx} = -2xy^3 - 4y \\
\frac{dy}{dx} = \frac{-2xy^3 - 4y}{3x^2y^2 + 4x}.
\]

11. \( x^2 - 4y^2 = 3x^3y^4 \)

\[
d \frac{d}{dx} (x^2 - 4y^2) = \frac{d}{dx} (3x^3y^4) \\
2x - 8y \frac{dy}{dx} = 9x^2y^4 + 3x^3 \cdot 4y^3 \frac{dy}{dx} \\
(-8y - 3x^3) \frac{dy}{dx} = 9x^2y^4 - 2x \\
\frac{dy}{dx} = \frac{9x^2y^4 - 2x}{8y + 12x^3y^3}.
\]

12. \( 9\sqrt{x} + 4y^3 = 2\sqrt{y} \)

\[
\frac{d}{dx} (9\sqrt{x} + 4y^3) = \frac{d}{dx} (2\sqrt{y}) \\
9 \frac{1}{2} x^{-1/2} + 12y^3 \frac{dy}{dx} = 2 \cdot \frac{1}{2} y^{-1/2} \cdot \frac{dy}{dx} \\
9 \frac{1}{2} x^{-1/2} = \frac{1}{2} y^{-1/2} \cdot \frac{dy}{dx} \\
\frac{dy}{dx} = \frac{9y^{1/2}}{2x^{1/2} (1 - 12y^{3/2})} = \frac{9\sqrt{y}}{2\sqrt{x}(1 - 12y^{3/2})}
\]

13. \( 2\sqrt{y - 1} = 9x^{2/3} + y \)

\[
\frac{d}{dx} [2(y - 1)^{1/2}] = \frac{d}{dx} (9x^{2/3} + y) \\
2 \cdot \frac{1}{2} (y - 1)^{-1/2} \frac{dy}{dx} = 6x^{-1/3} + \frac{dy}{dx} \\
[(y - 1)^{-1/2} - 1] \frac{dy}{dx} = 6x^{-1/3} \\
\frac{1 - \sqrt{y - 1}}{\sqrt{y - 1}} \frac{dy}{dx} = \frac{6}{x^{1/3}} \\
\frac{dy}{dx} = \frac{6\sqrt{y - 1}}{x^{1/3}(1 - \sqrt{y - 1})}
\]

14. \( x + 2y = y^{1/2} \)

\[
\frac{d}{dx} (x + 2y) = \frac{d}{dx} \left[ y^{1/2}(x - 3y) \right] \\
1 + 2 \frac{dy}{dx} = y^{1/2} \left( 1 - 3 \frac{dy}{dx} \right) \\
+ \frac{1}{2} (x - 3y) y^{-1/2} \frac{dy}{dx}
\]

\[
1 + 2 \frac{dy}{dx} = y^{1/2} - 3y^{1/2} \frac{dy}{dx} + \frac{1}{2} xy^{-1/2} \frac{dy}{dx} \\
- 3 \frac{y^{1/2}}{2} \frac{dy}{dx}
\]

\[
(2 + 3y^{1/2} - \frac{1}{2} xy^{-1/2} + \frac{3}{2} y^{1/2}) \frac{dy}{dx} = y^{1/2} - 1 \\
\frac{2y^{1/2}}{2y^{1/2}} \left( 2 + \frac{2y^{1/2} - \frac{1}{2} xy^{-1/2}}{2y^{1/2}} \right) \frac{dy}{dx} = y^{1/2} - 1 \\
\left( \frac{4y^{1/2} + 9y - x}{2y^{1/2}} \right) \frac{dy}{dx} = y^{1/2} - 1 \\
\frac{dy}{dx} = \frac{2y - 2y^{1/2}}{4y^{1/2} + 9y - x}.
\]
15. \( \frac{6 + 5x}{2 - 3y} = \frac{1}{5x} \)

\[
5x(6 + 5x) = 2 - 3y \\
30x + 25x^2 = 2 - 3y \\
\frac{d}{dx} (30x + 25x^2) = \frac{d}{dx} (2 - 3y) \\
30 + 50x = -3 \frac{dy}{dx} \\
-30 - 50x = \frac{dy}{dx}
\]

16. \( \ln(x + y) = 1 + x^2 + y^3 \)

\[
\frac{d}{dx} [\ln(x + y)] = \frac{d}{dx} (1 + x^2 + y^3) \\
\frac{1}{x + y} \cdot \frac{d}{dx} (x + y) = 2x + 3y^2 \cdot \frac{dy}{dx} \\
\frac{1}{x + y} (1 + \frac{dy}{dx}) = 2x + 3y^2 \cdot \frac{dy}{dx} \\
\left( \frac{1}{x + y} - 3y^2 \right) \frac{dy}{dx} = 2x - \frac{1}{x + y} \\
\frac{dy}{dx} = \frac{2x - \frac{1}{x + y}}{\frac{1}{x + y} - 3y^2} \\
= \frac{2x(x + y) - 1}{1 - 3y^2(x + y)} \\
= \frac{2x^2 + 2xy - 1}{1 - 3xy^2 - 3y^3} \\
= \frac{1 - 2x^2 - 2xy + 3xy^2 + 3y^3}{1 - 2x^2 - 2xy + 3xy^2 + 3y^3}
\]

17. \( \ln(xy + 1) = 2xy^3 + 4 \)

\[
\frac{d}{dx} [\ln(xy + 1)] = \frac{d}{dx} (2xy^3 + 4) \\
\frac{1}{xy + 1} \cdot \frac{d}{dx} (xy + 1) = 2y^3 + 2x \cdot 3y^2 \frac{dy}{dx} + \frac{d}{dx} (4) \\
\frac{1}{xy + 1} (y + x \frac{dy}{dx} + x \frac{dy}{dx} (1)) = 2y^3 + 6xy^2 \frac{dy}{dx} \\
\frac{y}{xy + 1} + \frac{x}{xy + 1} \cdot \frac{dy}{dx} = 2y^3 + 6xy^2 \frac{dy}{dx} \\
\left( \frac{x}{xy + 1} - 6xy^2 \right) \frac{dy}{dx} = 2y^3 - \frac{y}{xy + 1} \\
\frac{dy}{dx} = \frac{2y^3 - \frac{y}{xy + 1}}{\frac{x}{xy + 1} - 6xy^2} \\
= \frac{2y^3(xy + 1) - y}{x - 6xy^2(xy + 1)} \\
= \frac{2xy^4 + 2y^3 - y}{x - 6x^2y^3 - 6xy^2}
\]

18. \( \sqrt{2x} - 4y = -22 \), tangent line at \( (2, 3) \)

\[
\frac{d}{dx} (\sqrt{2x} - 4y) = \frac{d}{dx} (-22) \\
\frac{1}{2} (2x)^{-1/2} (2) - 4y \frac{dy}{dx} = 0 \\
(2x)^{-1/2} - 4y = 4x \frac{dy}{dx} \\
\frac{dy}{dx} = \frac{(2x)^{-1/2} - 4y}{4x} = \frac{\sqrt{2x} - 4y}{4x} \\
\text{At (2, 3),} \\
m = \frac{1}{2\sqrt{2}} - \frac{4 \cdot 3}{4 \cdot 2} = \frac{1}{2} - \frac{12}{8} = -\frac{23}{16}.
\]

The equation of the tangent line is

\[
y - y_1 = m(x - x_1) \\
y - 3 = -\frac{23}{16}(x - 2) \\
23x + 16y = 94.
\]

21. \( y = 8x^3 - 7x^2, \frac{dy}{dt} = 4, x = 2 \)

\[
\frac{dy}{dt} = \frac{d}{dt} (8x^3 - 7x^2) \\
= 24x^2 \frac{dx}{dt} - 14x \frac{dx}{dt} \\
= 24(2)^2 - 14(2) \frac{dx}{dt} \\
= 272
\]

22. \( y = 9 - 4x, \frac{dy}{dt} = -1, x = -3 \)

\[
\frac{dy}{dt} = \frac{d}{dt} (-4 + 3 + 2x) = (3 + 2x) \frac{dx}{dt} \\
= \frac{3}{3 + 2(-3)} \frac{dx}{dt} = \frac{3}{-3} \frac{dx}{dt} \\
= -30 \frac{dx}{dt} \\
\frac{dy}{dt} = \frac{-30}{9} = \frac{10}{3}
\]

23. \( y = \frac{1 + \sqrt{x}}{1 - \sqrt{x}}, \frac{dy}{dt} = -4, x = 4 \)

\[
\frac{dy}{dt} = \frac{d}{dt} \left[ \frac{1 + \sqrt{x}}{1 - \sqrt{x}} \right] \\
= \left(1 - \sqrt{x} \right) \frac{1}{2} \left( x^{-1/2} \right) - \left(1 + \sqrt{x} \right) \left( x^{-1/2} \right) \frac{d}{dx} \left( x^{-1/2} \right) \\
= \frac{(1 - 2) (1/2x) ((-4) - (1 + 2) (1/2x)) (-4)}{(1 - 2)^2} \\
= \frac{1 - 3}{1} = -2
\]
24. \(x^2 + 5y = 2; \frac{dx}{dt} = 1, x = 2, y = 0\)
\[x^2 + 5y = 2(x - 2y)\]
\[x^2 + 5y = 2x - 4y\]
\[9y = -x^2 + 2x\]
\[y = \frac{1}{9}(-x^2 + 2x)\]
\[\frac{dy}{dt} = \left(\frac{-2}{9} + \frac{2}{9}\right)\frac{dx}{dt}\]
\[= \left[\left(\frac{-2}{9}\right)(2) + \frac{2}{9}(1)\right]\]
\[= -\frac{4}{9} + \frac{2}{9} = -\frac{2}{9}\]

25. \(y = xe^{3x}; \frac{dx}{dt} = -2, x = 1\)
\[\frac{dy}{dt} = \frac{d}{dt}(xe^{3x})\]
\[= \frac{dx}{dt}e^{3x} + x\frac{d}{dt}(e^{3x})\]
\[= \frac{dx}{dt}e^{3x} + xe^{3x}\cdot 3\frac{dx}{dt}\]
\[= (1 + 3x)e^{3x}\frac{dx}{dt}\]
\[= (1 + 3\cdot 1)e^{3(1)}(-2) = -8e^3\]

26. \(y = \frac{1}{e^{x^2} + 1}; \frac{dx}{dt} = 3, x = 1\)
\[\frac{dy}{dt} = \frac{d}{dt}\left(\frac{1}{e^{x^2} + 1}\right)\]
\[= \frac{-1}{(e^{x^2} + 1)^2}\frac{d}{dt}(e^{x^2} + 1)\]
\[= \frac{-1}{(e^{x^2} + 1)^2}\left[e^{x^2}\cdot \frac{d}{dt}(x^2) + \frac{d}{dt}(1)\right]\]
\[= \frac{-1}{(e^{x^2} + 1)^2}\left[e^{x^2}\cdot 2x\frac{dx}{dt}\right]\]
\[= \frac{-1}{(e^{x^2} + 1)^2}\left[e\cdot 2\cdot 1\cdot 3\right]\]
\[= \frac{-6e}{(e + 1)^2}\]

28. \(y = 8 - x^2 + x^3, x = -1, \Delta x = 0.02\)
\[\frac{dy}{dx} = (-2x + 3x^2)\frac{dx}{dx} = (-2x + 3x^2)\Delta x\]
\[= [-2(-1) + 3(-1)^2](0.02)\]
\[= 0.1\]

29. \(y = \frac{3x - 7}{2x + 1}; x = 2, \Delta x = 0.003\)
\[\frac{dy}{dx} = \frac{(3)(2x + 1) - (2)(3x - 7)}{(2x + 1)^2}\]
\[dy = \frac{17}{(2x + 1)^2} dx\]
\[\approx \frac{17}{(2x + 1)^2} \cdot 0.003\]
\[= 0.000204\]

30. \(-12x + x^3 + y + y^2 = 4\)
\[\frac{dy}{dx}(-12x + x^3 + y + y^2) = \frac{d}{dx}(4)\]
\[-12 + 3x^2 + \frac{dy}{dx} + 2y \frac{dy}{dx} = 0\]
\[= (1 + 2y) \frac{dy}{dx} = 12 - 3x^2\]
\[= \frac{dy}{dx} = \frac{12 - 3x^2}{1 + 2y}\]

(a) If \(\frac{dy}{dx} = 0\),
\[12 - 3x^2 = 0\]
\[12 = 3x^2\]
\[\pm 2 = x.\]

\[x = 2:\]
\[-24 + 8 + y + y^2 = 4\]
\[y + y^2 = 20\]
\[y^2 + y - 20 = 0\]
\[(y + 5)(y - 4) = 0\]
\[y = -5 \text{ or } y = 4\]

\[(2, -5) \text{ and } (2, 4) \text{ are critical points.}\]

\[x = -2:\]
\[24 - 8 + y + y^2 = 4\]
\[y + y^2 = -12\]
\[y^2 + y + 12 = 0\]
\[y = \frac{-1 \pm \sqrt{1^2 - 48}}{2}\]

This leads to imaginary roots.
\[x = -2 \text{ does not produce critical points.}\]
(b) \( x \quad y_1 \quad y_2 \\
1.9 \quad -4.99 \quad 3.99 \\
2 \quad -5 \quad 4 \\
2.1 \quad -4.99 \quad 3.99 \\

The point \((2, -5)\) is a relative minimum. 
The point \((2, 4)\) is a relative maximum.

(c) There is no absolute maximum or minimum for \(x\) or \(y\).

32. (a) \( P(x) = -x^3 + 10x^2 - 12x \)
    \[ P'(x) = -3x^2 + 20x - 12 = 0 \]
    \[ 3x^2 - 20x + 12 = 0 \]
    \[ (3x - 2)(x - 6) = 0 \]
    \[ 3x - 2 = 0 \quad \text{or} \quad x - 6 = 0 \]
    \[ x = \frac{2}{3} \quad \text{or} \quad x = 6 \]
    \[ P''(x) = -6x + 20 \]
    \[ P'' \left( \frac{2}{3} \right) = 16, \]
    which implies that \(x = \frac{2}{3}\) is location of minimum.
    \[ P''(6) = -16, \]
    which implies that \(x = 6\) is location of maximum. 
Thus, 600 boxes will produce a maximum profit.

(b) Maximum profit \(P(6)\)

\[ \frac{1}{2} : 9 \quad 49 \]
\[ 2 : 1 \quad 49 \]
\[ 3 : 99 \quad 3 \]

33. Let \(x\) = the length and width of a side of the base;
    \(h\) = the height.

The volume is 32 m\(^3\); the base is square and there is no top. Find the height, length, and width for minimum surface area.

\[
\begin{align*}
\text{Volume} &= x^2 h \\
x^2 h &= 32 \\
\frac{h}{x^2} &= \frac{32}{x^2} \\
\text{Surface area} &= x^2 + 4xh \\
A &= x^2 + 4x \left( \frac{32}{x^2} \right) \\
&= x^2 + 128x^{-1} \\
A' &= 2x - 128x^{-2}
\end{align*}
\]

If \(A' = 0\),
\[
\begin{align*}
2x^3 - 128 &= 0 \\
x^3 &= 64 \\
x &= 4.
\end{align*}
\]

\[ A''(x) = 2 + 2(128)x^{-3} \]
\[ A''(4) = 6 > 0 \]

The minimum is at \(x = 4\), where
\[ h = \frac{32}{4^2} = 2. \]

The dimensions are 2 m by 4 m by 4 m.

34. \( V = \pi r^2 h = 40 \), so \( h = \frac{40}{\pi r^2} \).

\[
\begin{align*}
A &= 2\pi r^2 + 2\pi rh \\
&= 2\pi r^2 + 80 \frac{r}{\pi} \\
\text{Cost} &= C(r) = 4(2\pi r^2) + 3 \left( \frac{80}{r} \right) \\
&= 8\pi r^2 + \frac{240}{r} \\
C'(r) &= 16\pi r - \frac{240}{r^2} \\
16\pi r - \frac{240}{r^2} &= 0 \\
16\pi r^3 &= 240 \\
\frac{r^3}{\pi} &= \frac{15}{\pi} \\
r &\approx 1.684
\end{align*}
\]

\[ C''(r) = 16\pi + \frac{480}{r^3} > 0, \text{ so } r = 1.684 \text{ minimizes cost.} \]
\[ h = \frac{40}{\pi r^2} = \frac{40}{\pi(1.684)^2} = 4.490 \]

The radius should be 1.684 in. and the height should be 4.490 in.
35. Volume of cylinder = \( \pi r^2 h \)
   Surface area of cylinder open at one end = \( 2\pi rh + \pi r^2 \).

\[
V = \pi r^2 h = 27 \\
\frac{h}{\pi r^2} = \frac{27}{r^2} \\
A = 2\pi r \left( \frac{27}{r^2} \right) + \pi r^2 \\
= 54\pi r^{-1} + \pi r^2 \\
A' = -54\pi r^{-2} + 2\pi r
\]

If \( A' = 0 \),

\[
2\pi r = \frac{54\pi}{r^2} \\
r^3 = 27 \\
r = 3
\]

If \( r = 3 \),

\[
A'' = 108\pi r^{-3} + 2\pi > 0,
\]

so the value at \( r = 3 \) is a minimum.

For the minimum cost, the radius of the bottom should be 3 inches.

36. \( M = 180,000 \) cases sold per year
   \( k = \$12 \), cost to store 1 case for 1 yr
   \( f = \$20 \), fixed cost for order
   \( x \) = the number of cases per order

\[
x = \sqrt{\frac{2fM}{k}} \\
= \sqrt{\frac{2(20)(180,000)}{12}} \\
= \sqrt{3,000,000} \\
= 100\sqrt{60} \\
\approx 775
\]

The store should order 775 cases each time.

37. Here \( k = 0.15 \), \( M = 20,000 \), and \( f = 12 \). We have

\[
q = \sqrt{\frac{2fM}{k}} = \sqrt{\frac{2(12)(20,000)}{0.15}} \\
= \sqrt{3,200,000} \approx 1789
\]

Ordering 1789 rolls each time minimizes annual cost.

38. Use equation (3) from Section 6.3 with \( k = 2 \), \( M = 240,000 \), and \( f = 15 \).

\[
q = \sqrt{\frac{2fM}{k}} = \sqrt{\frac{2(15)(240,000)}{2}} \\
= \sqrt{3,600,000} \approx 1897.4
\]

\( T(1897) \approx 3794.7333 \) and \( T(1898) \approx 3794.7334 \).
Since \( T(1897) < T(1898) \), then the number of batches per year should be

\[
\frac{M}{q} = \frac{240,000}{1897} \approx 127.
\]

39. Use equation (3) from Section 6.3 with \( k = 1 \), \( M = 128,000 \), and \( f = 10 \).

\[
q = \sqrt{\frac{2fM}{k}} = \sqrt{\frac{2(10)(128,000)}{1}} \\
= \sqrt{2,560,000} = 1600
\]

The number of lots that should be produced annually is

\[
\frac{M}{q} = \frac{128,000}{1600} = 80.
\]

40. \( q = \frac{A}{p^k} \)

\[
dq = -k \frac{A}{p^{k+1}} \\
\frac{dE}{dp} = - \frac{p}{q} \cdot \frac{dq}{dp} \\
= \left( - \frac{p}{A} \right) \left( -k \right) \left( \frac{A}{p^{k+1}} \right) \\
= \left( \frac{p^{k+1}}{A} \right) \left( -k \right) \left( \frac{A}{p^{k+1}} \right) \\
= k
\]

The demand is elastic when \( k > 1 \) and inelastic when \( k < 1 \).

41. \( A = \pi r^2 \), \( \frac{dr}{dt} = 4 \) ft/min, \( r = 7 \) ft

\[
\frac{dA}{dt} = 2\pi r \frac{dr}{dt} \\
\frac{dA}{dt} = 2\pi(7)(4) \\
\frac{dA}{dt} = 56\pi
\]

The rate of change of the area is \( 56\pi \) ft\(^2\)/min.
42. \[ \frac{dx}{dt} = rx(N - x) \]
\[ = rxN - rx^2 \]
\[ \frac{d^2x}{dt^2} = rN \frac{dx}{dt} - 2rx \frac{dx}{dt} \]
\[ = r \frac{dx}{dt} (N - 2x) \]
\[ = r[xr(N - x)][N - 2x] \]
\[ \frac{d^2x}{dt^2} = 0 \text{ when } x = 0, x = N, \text{ or } x = \frac{N}{2}. \]

On \((0, \frac{N}{2})\), \(\frac{d^2x}{dt^2} > 0\); therefore, the curve is concave upward.

On \((\frac{N}{2}, N)\), \(\frac{d^2x}{dt^2} < 0\); therefore, the curve is concave downward.

Hence \(x = \frac{N}{2}\) is a point of inflection.

43. (a) 

(b) We use a graphing calculator to graph

\[ M'(t) = -0.4321173 + 0.1129024t - 0.0061518t^2 + 0.0001260t^3 - 0.000009825t^4 \]
on \([3, 51]\) by \([0, 0.3]\). We find the maximum value of \(M'(t)\) on this graph at about 15.41, or on about the 15th day.

44. (a) 

(b) To find where the maximum and minimum numbers occur, use a graphing calculator to locate any extreme points on the graph. One critical number is formed at about 87.78.

<table>
<thead>
<tr>
<th>(t)</th>
<th>(P(t))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>237.09</td>
</tr>
<tr>
<td>87.78</td>
<td>43.56</td>
</tr>
<tr>
<td>95</td>
<td>48.66</td>
</tr>
</tbody>
</table>

The maximum number of polygons is about 237 at birth. The minimum number is about 44.

45. Let \(x\) = the distance from the base of the ladder to the building; \(y\) = the height on the building at the top of the ladder.

\[ \frac{dy}{dt} = -2 \]
\[ 50^2 = x^2 + y^2 \]
\[ 0 = 2x \frac{dx}{dt} + 2y \frac{dy}{dt} \]
\[ \frac{dx}{dt} = -\frac{y}{x} \frac{dy}{dt} \]

When \(x = 30\), \(y = \sqrt{2500 - (30)^2} = 40\). So

\[ \frac{dx}{dt} = -\frac{40}{30}(-2) = \frac{80}{30} = \frac{8}{3} \]

The base of the ladder is slipping away from the building at a rate of \(\frac{8}{3}\) ft/min.

46. \(\frac{dV}{dt} = 0.9 \text{ ft}^3/\text{min}\)

Find \(\frac{dr}{dt}\) when \(r = 1.7\) ft

\[ V = \frac{\pi}{3} r^3 \]
\[ \frac{dV}{dt} = \frac{4}{3} \pi (3)r^2 \frac{dr}{dt} \]
\[ = 4\pi r^2 \frac{dr}{dt} \]
\[ 0.9 = 4\pi (1.7^2) \frac{dr}{dt} \]
\[ \frac{dr}{dt} = 0.9 \approx 0.0248 \]

The radius is changing at the rate of \(\frac{0.9}{11.56\pi} \approx 0.0248 \text{ ft/min}\).
47. Let \( x = \) one-half the width of the triangular cross section; 
\( h = \) the height of the water; 
\( V = \) the volume of the water.
\[
\frac{dV}{dt} = 3.5 \text{ ft}^3/\text{min}
\]
Find \( \frac{dh}{dt} \) when \( h = \frac{1}{3} \).

\[
V = \left( \text{Area of triangular side} \right) \left( \text{length} \right)
\]

Area of triangular cross section
\[
= \frac{1}{2}(\text{base})(\text{altitude})
= \frac{1}{2}(2x)(h) = xh
\]

By similar triangles, \( \frac{2x}{x} = \frac{2}{1} \), so \( x = h \).
\[
V = (xh)(4)
= h^2 \cdot 4
= 4h^2
\]
\[
\frac{dV}{dt} = 8h \frac{dh}{dt}
\]
\[
\frac{1}{8h} \frac{dV}{dt} = \frac{dh}{dt}
\]
\[
\frac{1}{8} \left( \frac{3.5}{4} \right) = \frac{dh}{dt}
\]
\[
\frac{dh}{dt} = \frac{21}{16} = 1.3125
\]

The depth of water is changing at the rate of 1.3125 ft/min.

48. \( V = \frac{4}{3} \pi r^3 \), \( r = 4 \text{ in.} \), \( \Delta r = 0.02 \text{ in.} \)
\[
dV = 4\pi r^2 \, dr
\]
\[
\Delta V \approx 4\pi r^2 \, \Delta r
= 4\pi(4^2)(0.02)
= 1.28\pi \text{ or about } 4.021
\]
The volume of the coating is 1.28\pi \text{ in}^3 \text{ or about } 4.021 \text{ in}^3.

49. \( A = s^2 \); \( s = 9.2 \), \( \Delta s = \pm 0.04 \)
\[
ds = 2s \, ds
\]
\[
\Delta A \approx 2s \, \Delta s
= 2(9.2)(\pm 0.04)
= \pm 0.736 \text{ in}^2
\]

50. \( V = l \cdot w \cdot h \)
\[
w = 4 + h
\]
\[
l + g = 130; \ g = 2(w + h)
\]
\[
l + 2(w + h) = 130
\]
\[
l + 2w + 2h = 130
\]
\[
l = 130 - 2w - 2h
= 130 - 2(4 + h) - 2h
= 122 - 4h
\]
\[\begin{align*}
V &= l \cdot w \cdot h \\
&= (122 - 4h)(4 + h)h \\
&= 488h + 106h^2 - 4h^3
\end{align*}\]
\[
\frac{dV}{dh} = 488 + 212h - 12h^2
\]

Set \( \frac{dV}{dh} = 0 \).
\[
488 + 212h - 12h^2 = 0
12h^2 - 212h - 488 = 0
3h^2 - 53h - 122 = 0
\]
\[
h = \frac{53 \pm \sqrt{2809 + 1464}}{6}
\approx -2.06 \text{ or } h = 19.73
\]
h can’t be negative, so
\[
h \approx 19.73.
\]
Thus,
\[
l = 122 - 4h
\approx 43.1
\]
The length that produces the maximum volume is about 43.1 inches.
51. We need to minimize \( y \). Note that \( x > 0 \).

\[
\frac{dy}{dx} = \frac{x}{8} - \frac{2}{x}
\]

Set the derivative equal to 0.

\[
x - \frac{2}{x} = 0
\]

\[
x = \frac{2}{x}
\]

\[
x^2 = 16
\]

\[
x = 4
\]

Since \( \lim_{x \to 0} y = \infty \), \( \lim_{x \to \infty} y = \infty \), and \( x = 4 \) is the only critical value in \((0, \infty)\), \( x = 4 \) produces a minimum value.

\[
y = \frac{4^2}{16} - 2 \ln 4 + \frac{1}{4} + 2 \ln 6
\]

\[
= 1.25 + 2(\ln 6 - \ln 4)
\]

\[
= 1.25 + 2 \ln 1.5
\]

The \( y \) coordinate of the Southern most point of the second boat’s path is \( 1.25 + 2 \ln 1.5 \).

52. Let \( x = \) width of play area;

\( y = \) length of play area.

An equation describing the amount of fencing is

\[
900 = 2x + y
\]

\[
y = 900 - 2x.
\]

Then,

\[
A = xy
\]

\[
A(x) = x(900 - 2x)
\]

\[
= 900x - 2x^2.
\]

If \( A'(x) = 900 - 4x = 0 \),

\[
x = 225.
\]

Then \( y = 900 - 2(225) = 450 \).

\( A''(x) = -4 < 0 \), so the area is maximized if the dimensions are 225 m by 450 m.

53. Distance on shore: \( 40 - x \) feet

Speed on shore: 5 feet per second

Distance in water: \( \sqrt{x^2 + 40^2} \) feet

Speed in water: 3 feet per second

The total travel time \( t \) is

\[
t = t_1 + t_2 = \frac{d_1}{v_1} + \frac{d_2}{v_2}.
\]

\[
t(x) = \frac{40-x}{5} + \frac{\sqrt{x^2 + 40^2}}{3}
\]

\[
= 8 - \frac{x}{5} + \frac{\sqrt{x^2 + 1600}}{3}
\]

\[
t'(x) = -\frac{1}{5} + \frac{1}{3} \frac{1}{2} (x^2 + 1600)^{-1/2} (2x)
\]

\[
= -\frac{1}{5} + \frac{x}{3 \sqrt{x^2 + 1600}}
\]

Minimize the travel time \( t(x) \). If \( t'(x) = 0 \):

\[
\frac{x}{3 \sqrt{x^2 + 1600}} = \frac{1}{5}
\]

\[
5x = 3 \sqrt{x^2 + 1600}
\]

\[
\frac{5x}{3} = \sqrt{x^2 + 1600}
\]

\[
\frac{25}{9} x^2 = x^2 + 1600
\]

\[
\frac{16}{9} x^2 = 1600
\]

\[
x^2 = \frac{1600 \cdot 9}{16}
\]

\[
x = \frac{40 \cdot 3}{4} = 30
\]

(Discard the negative solution.)

To minimize the time, he should walk \( 40 - x = 40 - 30 = 10 \) ft along the shore before paddling toward the desired destination. The minimum travel time is

\[
\frac{40 - 30}{5} + \frac{\sqrt{30^2 + 40^2}}{3} \approx 18.67 \text{ seconds.}
\]
54. Distance on shore: \(25 - x\) feet
   Speed on shore: 5 feet per second
   Distance in water: \(\sqrt{x^2 + 40^2}\) feet
   Speed in water: 3 feet per second

The total travel time \(t\) is
\[
t(x) = \frac{25 - x}{5} + \frac{\sqrt{x^2 + 40^2}}{3} \equiv 5 - \frac{x}{5} + \frac{\sqrt{x^2 + 1600}}{3}
\]
\[
t'(x) = -\frac{1}{5} + \frac{1}{3} \cdot \frac{1}{2} (x^2 + 1600)^{-1/2} (2x)
\]
\[
t'(x) = -\frac{1}{5} + \frac{x}{3\sqrt{x^2 + 1600}}
\]

Minimize the travel time \(t(x)\). If \(t'(x) = 0:\n\[
\frac{x}{3\sqrt{x^2 + 1600}} = \frac{1}{5}
\]
\[
5x = 3\sqrt{x^2 + 1600}
\]
\[
5x = \sqrt{x^2 + 1600}
\]
\[
\frac{25}{9} x^2 = x^2 + 1600
\]
\[
\frac{16}{9} x^2 = 1600
\]
\[
x^2 = \frac{1600 \cdot 9}{16}
\]
\[
x = \frac{40 \cdot 3}{4} = 30
\]

(Discard the negative solution.)

\(x = 30\) is impossible since the closest point on the shore to the desired destination is only 25 ft from where he is standing.

Check the endpoints.

<table>
<thead>
<tr>
<th>(x)</th>
<th>(t(x))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>18.33</td>
</tr>
<tr>
<td>25</td>
<td>15.72</td>
</tr>
</tbody>
</table>

The time is minimized when \(x = 25\).

\(25 - x = 25 - 25 = 0\) ft, so the mathematician should start paddling where he is standing.

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### Extended Application: A Total Cost Model for a Training Program

1. \(Z(m) = \frac{C_1}{m} + DmC_2 + DC_3 \left(\frac{m - 1}{2}\right)\)
2. \(Z'(m) = -\frac{C_1}{m^2} + \frac{DC_3}{m}\)
3. \(Z''(m) = 0\) when
   \[
   \frac{DC_3}{2} = \frac{C_1}{m^2}
   \]
   \[
   m^2 = \frac{2C_1}{DC_3}
   \]
   \[
   m = \sqrt{\frac{2C_1}{DC_3}}
   \]
4. \(3 < 3.33 < 4\)
   \(m^+ = 4\) and \(m^- = 3\)
5. \(Z(m) = \frac{C_1 + mDmC_2}{m} + \frac{DC_3 \left(\frac{m(m-1)}{2}\right)}{m}\)
   \[
   C_1 = 15,000; \quad D = 3, \quad t = 12,
   \]
   \[
   C_2 = 100; \quad C_3 = 900
   \]
   \[
   Z(m^+) = Z(4)
   \]
   \[
   = \frac{15,000}{4} + 3(12)(100) + 3(900) \left(\frac{4 - 1}{2}\right)
   \]
   \[
   = 3750 + 3600 + 4050
   \]
   \[
   = $11,400
   \]
   \[
   Z(m^-) = Z(3)
   \]
   \[
   = \frac{15,000}{3} + 3600 + 3 \left(\frac{900(3 - 1)}{2}\right)
   \]
   \[
   = 5000 + 3600 + 2700
   \]
   \[
   = $11,300
   \]
6. Since $Z(3) < Z(4)$, the optimal time interval is 3 months.

\[
N = mD \\
= 3 \cdot 3 \\
= 9
\]

There should be 9 trainees per batch.