Area of Triangles, Arc Length, and Sector Area

Area of a Triangle
The area \( K \) of Triangle ABC is given by each of the following formulas:

\[
K = \frac{1}{2} bc \sin A \\
K = \frac{1}{2} ac \sin B \\
K = \frac{1}{2} ab \sin C \\
K = \frac{1}{2} a^2 \frac{\sin B \sin C}{\sin A} \\
K = \frac{1}{2} b^2 \frac{\sin A \sin C}{\sin B} \\
K = \frac{1}{2} c^2 \frac{\sin A \sin B}{\sin C}
\]


Heron's
\[
K = \sqrt{s(s-a)(s-b)(s-c)}, \quad \text{where} \quad s = \frac{1}{2}(a+b+c)
\]

EXAMPLE 1 – Applying Area Formulas

Find the area of Triangle ABC given the following conditions:

a. \( A = 115^\circ, \ b = 5, \ c = 8 \)

\[
K = \frac{1}{2} bc \sin A \\
= \frac{1}{2} (5)(8) \sin 115^\circ \\
= 18.1
\]

A triangular garden has sides of lengths 20 m, 30 m, and 30 m.

\[
S = \frac{1}{2} (20 + 30 + 30) \\
= \frac{1}{2} (80) \\
= 40
\]

\[
K = \sqrt{40(40-20)(40-30)(40-30)} \\
= \sqrt{40 \cdot 20 \cdot 10 \cdot 10} \\
= \sqrt{800,000} \approx 282.8 \text{ m}^2
\]

Length of an Arc

\[
C = 2\pi r \\
S = \frac{\theta}{2\pi} \cdot 2\pi r \\
S = r \theta \\
\theta \text{ must be in radians!}
\]

Area of a Sector

\[
A = \pi r^2 \\
A_s = \frac{\theta}{2\pi} \cdot \pi r^2 \\
A_s = \frac{1}{2} r^2 \theta
\]

EXAMPLE 2 – Arc Length and Area of a Sector

a. Find the length of an arc of a circle with radius 21 m that subtends a central angle of \(15^\circ\) → \(\frac{15^\circ \cdot \pi}{180^\circ} = \frac{\pi}{12}\)

\[
S = r \theta = (21)\left(\frac{\pi}{12}\right) = \frac{7\pi}{4} \approx 5.5 \text{ m}
\]

b. A central angle, \( \theta \), in a circle of radius 9 m is subtended by an arc of length 12 m. Find \( \theta \) in radians.

\[
S = r \theta \\
12 = \frac{9 \cdot \theta}{4} \\
\theta = \frac{4}{3}
\]
Linear and Angular Speed

Suppose a point moves along a circle. Let \( s \) be the distance the point travels in time \( t \).

Linear Speed:
\[
v = \frac{s}{t} = \frac{\theta}{t}
\]

Angular Speed:
\[
\omega = \frac{\theta}{t}
\]

\( \theta \) must be in radians

Relationship between linear and angular velocities: \( \text{linear includes the radius} \)

**EXAMPLE 3 – Application**

a. A ceiling fan with 16 inch blades rotates at 45 rpm.

A. Find the angular speed in rad/sec

\[
\omega = \frac{\theta}{t} = \frac{45 \times 2\pi}{1\text{min}} \times \frac{1\text{min}}{60\ \text{sec}} = \frac{3\pi}{2} \text{ rad/sec} = 4.71 \text{ rad/sec}
\]

B. Find the linear speed in in/sec

\[
v = 16 \times 4.71 \text{ rad/sec} = 75.4 \text{ in/sec}
\]

b. A woman is riding a bicycle whose wheels are 30 inches in diameter. If the wheels rotate at 150 rpm, find the speed at which she is traveling in mi/hr.

\[
V = \frac{r\omega}{t} = \frac{15 \times 30\pi}{1\text{min}} \times \frac{60\text{ min}}{1\text{ hr}} \times \frac{1\text{ mi}}{63,360\text{ in}} = 13.4 \text{ mi/hr}
\]

r = 15

r = 15

300\pi

linear speed

V = r\omega = \frac{15 \times 300\pi}{1\text{min}} \times \frac{60\text{ min}}{1\text{ hr}} \times \frac{1\text{ mi}}{63,360\text{ in}} = 13.4 \text{ mi/hr}

r = 3.625

c. The circular blade on a saw has a diameter of 7.25 inches and rotates at 4800 revolutions per minute. Find the linear speed of the saw teeth (in feet per second) as they contact the wood being cut.

\[
v = \frac{r\omega}{t} = \frac{3.625 \text{ in} \times 9600\pi}{1\text{ min}} \times \frac{1\text{ min}}{60\text{ sec}} \times \frac{1\text{ ft}}{12\text{ in}}
\]

\[
= 151.8 \text{ ft/sec}
\]